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Semi-Stationary Equilibrium in Leader-Follower Games*

Ngo Van Long[†], *Koji Shimomura*[‡]

Résumé / Abstract

Nous démontrons que dans certains jeux de leader-followers, un état stationnaire n'existe qu'avec des prix fictifs non-stationnaires. Ce type d'équilibre est nommé « équilibre semi-stationnaire ». Les conclusions basées sur l'hypothèse que les prix fictifs sont stationnaires peuvent s'avérer fausses.

We show that in some leader-follower games, a steady-state equilibrium in quantities may exist only with non-stationary shadow prices. We call this type of equilibrium a semi-stationary steady state. Conclusions that are drawn on the assumption that a steady-state equilibrium has stationary shadow prices may turn out to be incorrect, because such a fully stationary steady state may not exist.

Mots Clés : Leader-follower, jeux dynamiques

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1 Introduction

The dynamics of redistributive taxation has received a great deal of interest among theorists. Quite often, the government is modelled as a Stackelberg leader that announces a time path of tax rates, and other economic agents such as workers and capitalists are modelled as followers that maximize their intertemporal utility, taking the announced time path of tax rates as given. One criticism that has been raised against this approach is that the government's solution is generally time-inconsistent. This criticism has turned out to be somewhat exaggerated, because recently it has been discovered that, depending on the specific structure of the model under consideration, an open-loop Stackelberg equilibrium may have the desired time-consistency property[1]. Another criticism of some analysis of open-loop Stackelberg redistributive taxation game is that the stability of steady-state equilibria is often assumed. Kemp, Long, and Shimomura (1993) have shown that around a steady state there may exist cycles, and cyclical redistributive taxation may be optimal.

In this paper, we raise a third point concerning the standard analysis of models of redistributive taxation. It is about the *assumption* that at a steady-state equilibrium of a redistributive game, all quantity variables *and all shadow prices* are stationary. This assumption has led to the claim that at a steady-state equilibrium, the tax rate on capital income must be zero, regardless of the elasticity of intertemporal substitution (see, for example, Judd (1985)). Our paper offers a counter-example which shows that (i) a steady-state equilibrium (in quantities) exists while the shadow prices are not stationary (we call this a "semi-stationary" state), (ii) at this steady-state equilibrium, the tax rate on capital income is non-zero, and (iii) in our example, there is no other steady-state equilibrium in quantities. Our counter-example, valid for open economies as well as closed economies, is important because it indicates that the standard short-cut of setting all equations of motions to zero can lead to incorrect conclusion about optimal tax in a steady state.

[1] See Xie (1997). The basic point in Xie's article is that a so-called "transversality condition" at the initial time of the control problem of the Stackelberg leader may turn out not to be a necessary condition, as was commonly claimed. For a more detailed discussion of this point, see Dockner, Jørgensen, Long, and Sorger (1999, Chapter 5), who also provide examples of the non-applicability of such "transversality conditions" in finite horizon and infinite horizon open-loop Stackelberg games.

The plan of our paper is as follows. In Section 2, we consider a model of redistributive taxation in a closed economy, and prove our basic results. In Section 3, we show that our arguments apply also to open economies. An alternative formulation of the redistributive game is presented in Section 4, where it is shown that the same results are obtained. Section 5 provides an intuitive explanation of our results. Some concluding remarks are offered in Section 6. The appendices settle some technical issues. In particular, Appendix C shows that our results are valid also when the model is set in discrete time.

2 Redistributive Taxation in a Closed Economy

We consider a model introduced by Judd (1985) in which there exists a government, and two types of infinitely-lived individuals, called workers and capitalists respectively. Each worker supplies one unit of labour, independently of the wage rate. They neither borrow nor lend: they consume all their current income. Capitalists do not work. They own capital, which they lend to competitive firms at the market interest rate $r(t)$. There is no depreciation. Firms use capital and labour to produce a final good which can be either consumed or invested. Constant returns to scale is assumed, and firms earn no profit. For notational simplicity, the number of each type of individual is set at unity, without loss of generality.

The gross income earned by the representative capitalist is $r(t)k(t)$ where $k(t) \geq 0$ is his stock of capital. His net income is $[1-\tau(t)]r(t)k(t)$ where $\tau(t)$ is the tax rate on capital income, and $1 \geq \tau(t) \geq 0$. Let $c(t)$ denote his rate of consumption, and let $z(t) = [1-\tau(t)]r(t)$. We assume that $z(t) \geq 0$. His stock of capital accumulates according to the equation

$$\dot{k}(t) = z(t)k(t) - c(t) \tag{1}$$

where $\dot{k}(t)$ is positive [respectively, negative] if consumption is less than [respectively, exceeds] net income. The capitalist has the utility function $u(c)$ which is increasing and strictly concave. We also assume that $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(0) = 0$. These assumptions ensure that it is never optimal to set $c = 0$. The capitalist takes the time path of $r(t)$ and $\tau(t)$ as given, and his optimization problem is to choose a time path $c(t)$ to maximize the integral of his stream of discounted utility

$$\int_0^\infty e^{-\rho t} u(c(t))dt$$

subject to (1) and $k(0) = k_0 > 0$ (given), where $\rho > 0$ is the rate at which utility is discounted. It is also understood that if $k(t) = 0$ then $c(t) = 0$. We impose either the constraint that, for all t

$$k(t) \geq 0 \tag{2}$$

or the constraint that

$$\lim_{t\to\infty} k(t) \exp\left[\int_{t_0}^{t} -z(\tau)d\tau\right] = 0. \tag{3}$$

The capitailist thus faces an optimal control problem[2]. Under our assumption about $u(c)$, it is well known that if $\tau(t) < 1$ and $r(t) > 0$ then $k(t)$ will always be positive along the optimal path.

Before solving the above problem, it is useful to point out that instead of (1) and (2) we may write the following intertemporal budget constraint

$$\int_{t_0}^{\infty} c(t) \exp\left[\int_{t_0}^{t} -z(\tau)d\tau\right] dt = k(t_0) \tag{4}$$

where $t_0 = 0$, but clearly t_0 can be any point of time. Note that we can integrate (1) to get

$$\lim_{t\to\infty} k(t) \exp\left[\int_{t_0}^{t} -z(\tau)d\tau\right] - k(t_0) = -\lim_{t\to\infty} \int_{t_0}^{t} c(s) \exp\left[\int_{t_0}^{s} -z(\tau)d\tau\right] ds \tag{5}$$

so (1) and (3) imply (4). Conversely, one can show that (4) implies (1) and (3).

2.1 The capitalist's consumption decision

The Hamiltonian for the capitalist's optimization problem is

$$H^c = u(c) + \psi[zk - c] \tag{6}$$

[2]See, for example, Leonard and Long (1992).

From this we obtain the necessary conditions

$$\frac{\partial H^c}{\partial c} = u'(c) - \psi = 0 \tag{7}$$

$$\dot{\psi} = \rho\psi - \frac{\partial H^c}{\partial k} = \psi(\rho - z) \tag{8}$$

$$\dot{k} = \frac{\partial H^c}{\partial \psi} = zk - c \tag{9}$$

and the transversality condition is

$$\lim_{t\to\infty} e^{-\rho t} u'(c(t))k(t) = 0. \tag{10}$$

Differentiating (7) with respect to t we get

$$u''(c)\dot{c} = \dot{\psi} = \psi(\rho - z) = u'(c)(\rho - z)$$

Therefore conditions (7) and (8) can be combined into a single condition

$$\dot{c} = \frac{c}{\beta(c)}(z - \rho) \tag{11}$$

where by definition $\beta(c) = -u''(c)c/u'(c) > 0$ is the elasticity of marginal utility. The intertemporal elasticity of substitution is defined as $1/\beta(c)$. In what follows, we assume that $\beta(c)$ is a constant, denoted by β.

Remark 1:

Under the assumption that β is a constant, we can integrate (11) to get

$$c(t) = c(t_0)\exp\left[\beta^{-1}\int_{t_0}^{t}(z(\tau) - \rho)d\tau\right] \tag{12}$$

Substituting (12) into (4), we get

$$c(t_0) = \frac{k(t_0)}{\int_{t_0}^{\infty}\exp\{\int_{t_0}^{\tau}[z(s)(\beta^{-1} - 1) - (\rho/\beta)]ds\}d\tau} \tag{13}$$

Thus current consumption depends on future tax rates if $\beta \neq 1$. However, if $\beta = 1$, which is the case of a logarithmic utility function, then (13) becomes

$$c(t_0) = \rho k(t_0). \tag{14}$$

2.2 The government's optimization problem

Under competitive conditions and given the assumption that there is no depreciation, the interest rate is the same as the marginal product of capital, $f'(k)$. The wage rate is $w(t) = f(k) - r(t)k(t)$.

The government's role is assumed to be purely redistributive: the capital income tax revenue $\tau(t)r(t)k(t)$ is given to the representative worker, who consumes both $\tau(t)r(t)k(t)$ and his wage income $w(t)$. Since

$$\tau(t)r(r)k(t) = r(t)k(t) - z(t)k(t) = [f(k(t)) - w(t)] - z(t)k(t)$$

the consumption of the representative worker is

$$c_w(t) = f(k(t)) - z(t)k(t). \tag{15}$$

The utility of the worker is $v(c_w)$ where $v(.)$ is strictly concave and increasing.

The government takes the pair of differential equations (9) and (11), together with the initial condition $k(0) = k_0$ and the transversality condition (10), as representing the representative capitalist's "reaction function" to the government's announced time path $z(t)$. The government's optimization problem consists of choosing a time path $z(t)$ to maximize the integral of discounted flow of the weighted sum of utility $\gamma v(c_w(t)) + u(c(t))$:

$$\max \int_0^\infty e^{-\rho t}\{\gamma v(c_w(t)) + u(c(t))\}dt \tag{16}$$

subject to (15) and

$$\dot{k}(t) = z(t)k(t) - c(t) \tag{17}$$

$$\dot{c}(t) = \frac{c(t)}{\beta}\{z(t) - \rho\} \tag{18}$$

$$k(0) = k_0 \tag{19}$$

$$\lim_{t\to\infty} e^{-\rho t}u'(c(t))k(t) = 0 \tag{20}$$

where $\gamma > 0$ is the weight given to the utility of the worker.

Notice that in this formulation, the capital stock is not the only state variable: we treat c as a state variable, too. (This is the standard approach, see Judd (1985), Kemp, Long and Shimomura (1993), among others.) The Hamiltonian for this problem is

$$H = \gamma v(f(k) - zk) + u(c) + p[zk - c] + qc\left[\frac{z-\rho}{\beta}\right] \tag{21}$$

and the necessary conditions are

$$\frac{\partial H}{\partial z} = -\gamma k v' + pk + qc/\beta = 0 \tag{22}$$

$$\dot{p} = \rho p - \frac{\partial H}{\partial k} = \rho p - zp - \gamma(f' - z)v' \tag{23}$$

$$\dot{q} = \rho q - \frac{\partial H}{\partial c} = \rho q - q\left[\frac{z-\rho}{\beta}\right] - u'(c) + p \tag{24}$$

together with (17), (18), (19), (20), and the transversality conditions

$$\lim_{t\to\infty} e^{-\rho t} pk = \lim_{t\to\infty} e^{-\rho t} qc = 0 \tag{25}$$

2.3 The optimal capital income tax rate

We want to find the optimal capital income tax rate in a steady state, defined as the situation where the state variables remain constant for ever, i.e. $\dot{c} = \dot{k} = 0$ for all t. Suppose we follows the standard procedure of finding a steady state by equating to zero the time derivatives of all the state variables *and all the associated shadow prices.* The steady state condition $\dot{c} = 0$ implies $z = \rho$ in the steady state. Using this result to simplify (23) and then *equating* $\dot{p}$ *to zero* gives us $f' = z$. But $f' = r$, therefore $z = r$, implying that in a steady state where the time derivatives of all shadow prices are equated to zero, the *tax rate on capital income is zero*, and the steady state capital stock k^* satisfies the *modified golden rule* property

$$f'(k^*) = \rho. \tag{26}$$

Since $\dot{k} = 0$, $c^* = \rho k^*$. Substitute this into (22) to get

$$k^*[p^* + \rho(q^*/\beta) - \gamma v'(f(k^*) - \rho k^*)] = 0 \tag{27}$$

Since we have set $\dot{p} = 0$, equation (27) holds at a steady state only if $\dot{q} = 0$, too.This in turn implies, via (24), that

$$\rho q^* + p^* = u'(\rho k^*) \tag{28}$$

Since k^* has been determined by (26), the two equations (27) and (28) determine the pair (q^*, p^*). It can be easily verified that q^* satisfies the following condition:

$$u'(\rho k^*) - \gamma v'(f(k^*) - \rho k^*) = \rho q^* \left[\frac{1}{\beta} - 1\right] \tag{29}$$

This indicates that the gap between the two weighted marginal utilities is generally non-zero.

It is tempting to conclude that in this model, at the optimal steady state, the optimal tax rate on capital income is zero. However, such a conclusion seems to be too hasty. Let us consider the case where the utility function $u(.)$ is logarithmic: $u(c) = \ln c$. In this case, $\beta = 1$, and $u'(\rho k^*) = 1/(\rho k^*)$; hence we can substitute (28) into (27) to get

$$k^*\{[1/(\rho k^*)] - \gamma v'(f(k^*) - \rho k^*)\} = 0 \tag{30}$$

In general it is not possible for k^* to satisfy both this equation and equation (26). It follows that if $u(c) = \ln c$ then a steady state where all shadow prices are stationary *does not exist* (generically). This is not a negative result, however, for, as will be shown below, there exists a steady state $(\hat{k}, \hat{c})$ where all state variables are stationary while the *shadow prices are not stationary.* At this steady state, however, the capital stock $\hat{k}$ *does not satisfy the modified golden rule* (26) and the *optimal tax on capital income is non-zero.*

First, setting $\dot{c} = 0$, we get

$$z = \rho \tag{31}$$

Next, setting $\dot{k} = 0$, we get

$$\hat{c} = \rho \hat{k} \tag{32}$$

Substituting these results into (22), with $\beta = 1$, we get

$$\hat{k}[p(t) + \rho q(t) - \gamma v'(f(\hat{k}) - \rho \hat{k})] = 0 \tag{33}$$

Since we seek a $\hat{k}$ which is positive and stationary, (33) implies

$$p(t) + \rho q(t) = \gamma v'[f(\hat{k}) - \rho \hat{k}] \tag{34}$$

and it follows that $\dot{p}(t) + \rho \dot{q}(t) = 0$. Multipling equation (24) by ρ, adding it to equation (23), and equating the result to zero, we get

$$\dot{p}(t) + \rho \dot{q}(t) = \gamma[2\rho - f'(\hat{k})]v'(f(\hat{k}) - \rho \hat{k}) - (1/\hat{k}) = 0 \tag{35}$$

This equation determines $\hat{k}$.

To summarize, when the utility function $u(c)$ is logarithmic, there exists a steady state pair $(\hat{k}, \hat{c})$ with non-stationary shadow prices $(p(t), q(t))$ where $\hat{k}$ is determined by

$$\gamma[2\rho - f'(\hat{k})]v'[f(\hat{k}) - \rho\hat{k}] - (1/\hat{k}) = 0 \tag{36}$$

and $\hat{c} = \rho\hat{k}$, while $p(t) + \rho q(t)$ is given by (34). (If the system begins at $(\hat{k}, \hat{c})$ then it will remain there, while the shadow prices will be changing continuously, with $p(t) = p(0) - \gamma[f'(\hat{k}) - \rho]v'[f(\hat{k}) - \rho\hat{k}]t$, where $p(0)$ is arbitrary.) Note that equation (36) can be written as:

$$u'(\rho\hat{k}) - \gamma v'[f(\hat{k}) - \rho\hat{k}] = \frac{1}{\rho}\left[\rho - f'(\hat{k})\right]\gamma v'[f(\hat{k}) - \rho\hat{k}] \tag{37}$$

This shows that the gap between the two weighted marginal utilities is generally non-zero. It is positive if and only if $f'(\hat{k}) < \rho$.

When the utility function $u(c)$ is logarithmic, the optimal tax on capital in the steady state is

$$\hat{\tau} = f'(\hat{k}) - z = f'(\hat{k}) - \rho = \rho - \frac{1}{\gamma\hat{k}v'[f(\hat{k}) - \rho\hat{k}]} \tag{38}$$

which can be positive or negative. From (38) and (37), $\hat{\tau}$ is negative if and only if $u'(\rho\hat{k})$ exceeds $\gamma v'[f(\hat{k}) - \rho\hat{k}]$.

3 Redistributive Taxation in an Open Economy

In this section, we seek to show that the basic results of the preceding section also applies to open economies. To do this in the simplest way, we consider a small open economy producing two goods: good 1, which is an investment good, and good 2, which is a consumption good. Their outputs are denoted by X_1 and X_2 respectively. The production functions are

$$X_i = F_i(K_i, L_i), \qquad i = 1, 2$$

where each $F_i(K_i, L_i)$ is homogenous of degree one, strictly quasi-concave, and increasing in each argument. To fix ideas, assume that good 1 is labor intensive, though this is not essential for our argument. Let π denote the (time-independent) price of the consumption good in terms of the investment good, which serves as the numeraire. The small open economy takes π as exogenously given. Perfect competition is assumed to prevail in the product markets and factor markets.

It is well known that the above assumptions allow us to obtain the GNP function

$$y = g(k, \pi)$$

where y is aggregate output in terms of the numeraire good, and $k(t)$ is the economy's stock of capital per worker. Let us list below the properties of the function $g(k, \pi)$. Given π, there exist two positive numbers $k_A(\pi)$ and $k_B(\pi)$, where $k_A(\pi) < k_B(\pi)$, such that (i) if $k \leq k_A(\pi)$, then the economy specializes in the production of good 1, and $g(k, \pi) = f_1(k)$; (ii) if $k \geq k_B(\pi)$, then the economy specializes in the production of good 2, and $g(k, \pi) = \pi f_2(k)$; (iii) if $k_A(\pi) < k < k_B(\pi)$, then the economy produces both goods, and aggregate output is

$$g(k, \pi) = f_1(k_A(\pi)) + (k - k_A(\pi)) f_1'(k_A(\pi))$$

Over the range of diversification described in (iii), the marginal product of capital is constant, and given by:

$$\frac{\partial g}{\partial k} = f_1'(k_A(\pi)).$$

In what follows, since π is assumed to be constant, we will suppress the dependence of y on π and write $f(k)$ as a short-hand expression for $g(k, \pi)$. Note that the function $f(k)$ is continuous, increasing, and concave, with strict concavity holding for $k \leq k_A(\pi)$, and for $k \geq k_B(\pi)$. For $k_A(\pi) < k < k_B(\pi)$, the function $f(k)$ is linear.

It is clear that the model and the analysis in section 2 can be adapted to our open economy. The minor changes that must be made are listed below.

1. The accumulation equation (1) becomes

$$\dot{k}(t) = [1 - \tau(t)] r(t) k(t) - \pi c(t) \tag{39}$$

2. The Hamiltonian (6) becomes

$$H^c = u(c) + \psi[zk - \pi c] \tag{40}$$

3. The consumption of the representative worker, instead of being (15), is now

$$c_w(t) = [f(k(t)) - z(t)k(t)]\pi^{-1}. \tag{41}$$

It follows that the equations (16) to (38) must be modified, but only with minimal changes. The essential results of section 2 carry over to the small open economy.

4 An Alternative Approach

In the preceding section we showed that when the capitalist's utility function is $u(c) = \ln c$, then a steady state with stationary shadow prices (in addition to stationary state variables) does not exist (generically), and that there exists a steady state with non-stationary shadow prices. At this steady state, the optimal tax on capital is non-zero. In deriving those conclusions, we have made used of the standard formulation of the government's optimization problem: we have introduced $c(t)$ as an additional state variable. In the present section, we show that the same results can be obtained using an alternative formulation.This approach consists of representing the capitalist's optimal consumption rule in a close form, which is possible when $u(c) = \ln c$.

Instead of representing the capitalist's optimal paths by the pair of differential equations (9) and (11), we note that in the case $\beta = 1$ (i.e., utility is logarithmic), using (3), we can represent these paths by the following pair of equations:

$$c(t) = \rho k(t) \tag{42}$$

and

$$\dot{k}(t) = z(t)k(t) - \rho k(t) \tag{43}$$

(In fact, this was shown in Remark 1 of subsection 2.1. Note also that if we differentiate (42) with respect to t, we get

$$\dot{c}(t) = \rho\dot{k}(t) = \rho[z(t)k(t) - \rho k(t)] = \rho[z(t)k(t) - c(t)]$$

which is (11) with $\beta = 1$.)

The representation (42)-(43) highlights the fact that when $\beta = 1$, the capitalist's current consumption $c(t)$ depends only on his current capital stock $k(t)$, and is independent of current and future tax rates. His current saving, $z(t)k(t) - c(t)$, does depends on the current tax rate.This fact is only implicit in the representation using (9) and (11).

Remark 2: If we do not use (3), we still can get the same result. With $\beta = 1$, we get from ((9)),

$$z(t) = \rho + \frac{\dot{c}(t)}{c(t)}$$

hence (11) yields

$$\frac{\dot{k}(t)}{k(t)} = \rho - \frac{\dot{c}(t)}{c(t)} + \frac{c(t)}{k(t)}$$

Define $\theta(t) = k(t)/c(t)$. Then

$$\frac{\dot{\theta}(t)}{\theta(t)} = \rho - \frac{1}{\theta(t)} \tag{44}$$

This equation, and the condition

$$\lim_{t \to \theta} \theta(t) e^{-\rho t} = 0 \tag{45}$$

which is obtained from (10), give $\theta(t) = \rho$ as the only solution that satisfies (45).

Using (42)-(43), the optimization problem of the government can be re-written as

$$\max \int_0^\infty e^{-\rho t} \{\gamma v[f(k(t)) - z(t)k(t)] + \ln(\rho k(t))\} dt$$

subject to (43) and $k(0) = k_0$.

The Hamiltonian is

$$\tilde{H} = \gamma v[f(k(t)) - z(t)k(t)] + \ln(\rho k(t)) + \psi[z(t)k(t) - \rho k(t)] \tag{46}$$

and the first order conditions are (43)

$$\frac{\partial \tilde{H}}{\partial z} = -\gamma k v' + \psi k = 0 \tag{47}$$

and

$$\dot{\psi} = \rho\psi - \frac{\partial \tilde{H}}{\partial k} = z\psi - \frac{1}{k} - \gamma[f'(k) - z]v' \tag{48}$$

At a steady state with $\dot{k} = \dot{\psi} = 0$, we have

$$z = \rho$$

and

$$2\rho - f'(k) = \frac{1}{\gamma k v'} \tag{49}$$

Equation (49) is identical to (35). It follows that the steady state of this problem is identical to that given by (35).

The approach taken in this section, while confirming our results found in the preceding section, has several advantages: (i) simplicity, (ii) sufficiency is ensured because the maximized Hamiltonian is concave in the state variable[3], and (iii) it is transparent that the solution is time-consistent.

5 An Intuitive Explanation

While the main purpose of this paper is to show that, in some model formulations, the unique steady state in quantities can be obtained only by allowing shadow prices to be non-stationary, we feel that it may be useful to provide some economic explanation. Why is it that, if the representative capitalist's utility function is logarithmic ($\beta = 1$), the optimal tax on capital income in the steady state is non-zero, while if their utility function has an elasticity of marginal utility $\beta \neq 1$, the optimal tax is zero in the steady state?

The reason is that if $\beta = 1$ (i.e., if the representative capitalist has a logarithmic utility function), then the income effect and the substitution effect (on his current consumption) of a change in capital income tax cancel each other out, and his optimal current consumption[4] is simply $\rho a(t)$ where $a(t)$ denotes the current level of his wealth (which, in our model, is the same as his capital stock $k(t)$). This means that $c(t)$ and $k(t)$ cannot be

[3] See the Appendix.

[4] The Hamilton-Jacobi-Bellman equation for the capitalist's optimization problem is

$$\rho V(t,a) - V_t(t,a) = \max\{\ln c + V_a(t,a)[z(t)a - c]\}$$

and it can be verified that the function

$$V(t,a) = \int_t^{\infty} \left\{ e^{-\rho(s-t)} \int_t^s [z(\tau) - \rho] d\tau \right\} ds + \frac{\ln(\rho a)}{\rho}$$

satisfies the above functional equation and that the optimal control rule is $c(t) = \rho a(t)$.

separated by the only policy instrument that we assume is available to the government, i.e,. the time path of the tax rate $\tau(.)$. The government can use this policy instrument to influence the time path $k(.)$, but it cannot break the relationship $c(t) = \rho k(t)$. If $\beta \neq 1$, the situation is quite different: there is no such fixed, tax-independent relationship[5]. Thus, if $\beta = 1$, the government is much handicapped, and it is no longer optimal to try to achieve the golden rule $f'(k) = \rho$ in the long run.

The above explanation raises an interesting question: given $\beta = 1$, what additional policy instrument would ensure that the optimal redistributive taxation implies a zero tax rate on capital income in the steady state? The key to the answer is that the nexus $c = \rho k$ must be broken. And this can be done if another asset is introduced into the economy, say government bonds, denoted by b.(If b is negative, this means the government is the net creditor.) For then, the representative capitalist's wealth, denoted by a, is the sum of k and b, and his optimal consumption rule is $c(t) = \rho[k(t) + b(t)]$, and it is no longer true that $c(t) = \rho k(t)$. It is shown in the Appendix that with this additional policy instrument, in the long run the optimal tax on capital income is zero, and the golden rule $f'(k) = \rho$ is restored. (If $b < 0$ in the steady state, which is the case if γ is sufficiently great, this means the government in effect owns part of the capital stock, and the rental income on that part is distributed to the workers).

6 Concluding Remarks

We have shown that it may be incorrect to characterize the properties of a steady state by setting the motion of all shadow prices equal to zero. Such a procedure can lead to policy recommendations that are unwarranted. Using the standard representation of the representative capitalist's "reaction function", we have also demonstrated that, in the case where the capitalist's utility function is logarithmic, the optimal tax on capital income in a steady state (with non-stationary shadow prices) is not zero, and there does not exist a steady state with stationary shadow prices. The steady state we found can

[5]Remark 1 in subsection 2.1 implies that, for all t,

$$c(t) = \frac{a(t)}{\int_t^\infty \exp\{\int_t^\tau z(s)(1/\beta - 1) - \rho/\beta ds\} d\tau}$$

Thus current consumption depends on future tax rates if $\beta \neq 1$.

also be obtained using a closed form representation of the capitalist's optimal consumption rule. Our results apply to both closed and open economies.

Another related issue in the taxation literature is whether government policies, obtained by solving an optimal control problem, is time inconsistent. Xie (1997) has shown that if the production function is linear in capital and the utility function is logarithmic, then the optimal tax path is time-consistent. In our model, even though the production function is not linear, the solution is also time-consistent as long as the utility function is logarithmic.

APPENDIX

A.Proof that the maximized Hamiltonian in Section 3 is concave in the state variable.

From (47) we get

$$f(k) - zk = v'^{-1}(\psi)$$

and hence $zk = f(k) - v'^{-1}(\psi)$. The maximized Hamiltonian is

$$H^m = \ln(\rho k) + \gamma v[v'^{-1}(\psi)] + \psi[f(k) - v'^{-1}(\psi)] - \psi\rho k$$

which is concave in k.

B. Steady-state when government bonds exist

Let $b(t)$ denote the stock of government bonds. A government bond is a piece of paper that entitles its holder to earn the market interest rate $r(t)$, and to exchange it, whenever he wishes to, with a unit of capital. Initially $b(0) = 0$. Let $T(t)$ be the lump-sum transfer to workers. The government's budget constraint is

$$\dot{b}(t) + \tau(t)[r(t)k(t) + r(t)b(t)] = r(t)b(t) + T(t)$$

(The right-hand side is the government disbursements, and the left-hand side is the government receipts, which consist of tax revenue and proceeds from the sale of additional bonds). Let $a(t) \equiv k(t) + b(t)$ be the total wealth of the representative capitalist. He is indifferent between using his savings to acquire additional government bonds and using them to build up his stock of capital goods. His optimal consumption strategy must solve

$$\max_c \int_0^\infty e^{-\rho t} \ln(c) dt$$

subject to

$$\dot{a}(t) = z(t)a(t) - c(t), \qquad a(t) \geq 0,$$

and $a(0) = a_0 = k_0$. This yields the optimal consumption rule

$$c(t) = \rho a(t).$$

It follows that

$$\dot{a}(t) = [z(t) - \rho]a(t) = [z(t) - \rho][k(t) + b(t)]$$

The consumption of the representative worker is $c_w(t) = w(t) + T(t) = f(k(t)) - r(t)k(t) + T(t)$. Using the government budget constraint, we can write

$$c_w(t) = f(k(t)) - z(t)[k(t) + b(t)] + \dot{b}(t)$$

Let $x(t) \equiv \dot{b}(t)$ be a control variable for the government. Adopting the approach used in section 4, we can write the optimization problem for the government as

$$\max \int_0^\infty e^{-\rho t}\{\gamma v(f(k) - z(k+b) + x) + \ln[\rho(k+b)]\}dt$$

subject to

$$\dot{b} = x$$

$$\dot{k} = \dot{a} - \dot{b} = (z - \rho)(k + b) - x$$

and $k(0) = k_0 > 0$, $b(0) = 0$.

The Hamiltonian is

$$H = \gamma v(f(k) - z(k+b) + x) + \ln[\rho(k+b)] + \psi_k[(z-\rho)(k+b) - x] + \psi_b x$$

From this we obtain

$$\frac{\partial H}{\partial z} = (k+b)[\psi_k - \gamma v'] = 0$$

$$\frac{\partial H}{\partial x} = \gamma v' - \psi_k + \psi_b = 0$$

$$\dot{\psi}_k = \rho\psi_k - (f' - z)\gamma v' - \frac{1}{k+b} + \psi_k(z - \rho)$$

$$\dot{\psi}_b = \rho\psi_b + z\gamma v' - \frac{1}{k+b}$$

It is straightforward to show that at a steady state with $\dot{\psi}_k = \dot{\psi}_b = \dot{k} = \dot{b} = 0$, we must have $z = \rho = f'(k^*)$, and b^* is given by

$$\gamma v'(f(k^*) - \rho(k^* + b^*)) = \frac{1}{\rho(k^* + b^*)} \tag{50}$$

Thus, in the presence of government bonds, the optimal tax on capital income is zero at the steady state. Notice that from (50), if γ is sufficiently great, then b^* is negative, implying that the government is a net creditor. One can consider, as an illustration, the following scenario: suppose that initially the economy has the capital stock $k(0) = k^*$ where $f'(k^*) = \rho$, and $b(0) = 0$. Then the government imposes a tax $\tau > 0$ so that the after-tax income of the representative capitalist is less than ρk^*. Since the capitalist's optimal consumption rule is $c = \rho k$, to satisfy this consumption and to pay the income tax, he must "eat up" part of his capital stock, if the government cannot be a creditor. But since we allow b to be negative, the government can offer the capitalist to pay part of the tax in the form of I.O.U.s, and there need not be any fall in k. This means that part of the physical capital stock k becomes effectively owned by the government. At a steady state with a negative b^*, even though the tax rate is zero, the government is able to make lump-sum payments to the workers, financed by the interest income it received from the capitalists.

C. A discrete time formulation

In this Appendix, we show that the results of Section 2 can also be obtained in a discrete time model. Capitalists maximize

$$\sum_{t=0}^{\infty} \delta^t u(c_t) \tag{51}$$

subject to

$$c_t + k_{t+1} = (1 - \tau_t) r_t k_t \tag{52}$$

where r_t is the gross rate of return, $0 < \delta < 1$ is the discount factor, and k_0 is given. This gives the Euler equation

$$u'(c_t) = \delta u'(c_{t+1})(1 - \tau_{t+1}) r_{t+1} \tag{53}$$

and the usual transversality condition. Workers consume

$$c_{wt} = w_t + \tau_t r_t k_t \tag{54}$$

The resource constraint is

$$c_t + c_{wt} + k_{t+1} = F(k_t) \tag{55}$$

where $F(k)$ is the gross output (i.e., output plus the depreciated stock of capital). Multiply (52) by $\delta^t u'(c_t)$, then sum from $t = 0$ to ∞, and using (53) and the transversality condition to get the implementability condition:

$$\sum_{t=0}^{\infty} \delta^t u'(c_t) c_t = u'(c_0)(1 - \tau_0) f'(k_0) k_0 \tag{56}$$

From (53) and (52),

$$c_t + k_{t+1} = \frac{k_t u'(c_{t-1})}{\delta u'(c_t)} \tag{57}$$

The government seeks to maximize

$$\sum_{t=0}^{\infty} \delta^t \left[u(c_t) + \gamma v(c_{wt}) \right] \tag{58}$$

subject to (55), (56) and (57). We form the Lagrangian and let μ_t, θ, and λ_t be the Lagrange multipliers corresponding to (55), (56) and (57). Differentiating the Lagrangian with respect to k_{t+1}, c_t, and c_{wt}, we obtain the following first order conditions

$$\lambda_t - \mu_t = \lambda_{t+1} \left[\frac{u'(c_t)}{\delta u'(c_{t+1})} \right] - \mu_{t+1} F'(k_{t+1}) \tag{59}$$

$$\delta^t u'(c_t) = \theta \delta^t \left[c_t u''(c_t) + u'(c_t) \right] - \lambda_t + \mu_t - \lambda_t \frac{k_t u'(c_{t-1}) u''(c_t)}{\delta \left[u'(c_t) \right]^2} + \lambda_{t+1} \frac{k_{t+1} u''(c_{t+1})}{\delta u'(c_{t+2})} \tag{60}$$

$$\delta^t \gamma v'(c_{wt}) = \mu_t \tag{61}$$

Now let us consider a semi-stationary state. That is, we let $c_t = \bar{c}$, $c_{wt} = \bar{c}_w$ and $k_t = \bar{k}$. From (52), we then have

$$\frac{\mu_t}{\delta^t} = \mu_0 = \gamma v'(\bar{c}_{wt}) \tag{62}$$

Notice that while μ_t/δ^t is a constant, it is not true that λ_t/δ^t must also be a constant. Equation (59) yields, in a semi-stationary state, a first order difference equation in the discounted multiplier λ_t/δ^t :

$$\frac{\lambda_{t+1}}{\delta^{t+1}} - \frac{\lambda_t}{\delta^t} = \mu_0 \left[\delta F'(\bar{k}) - 1\right] \tag{63}$$

From (60), in a semi-stationary state,

$$u'(\bar{c}) - \gamma v'(\bar{c}_{wt}) - \left[\bar{c}u''(\bar{c}) + u'(\bar{c})\right]\theta = \frac{\lambda_t}{\delta^t}\left[1 + \frac{\bar{k}u''(\bar{c})}{\delta u'(\bar{c})}\right] + \frac{\lambda_{t+1}}{\delta^{t+1}}\left[\frac{\bar{k}u''(\bar{c})}{u'(\bar{c})}\right] \tag{64}$$

Let us define ρ by $\delta = 1/(1+\rho)$. Then, in a semi-stationary state, (57) yields

$$\bar{c} = \rho\bar{k}. \tag{65}$$

Then the right-hand side of (64) becomes

$$RHS = \left[\frac{\lambda_{t+1}}{\delta^{t+1}} - \frac{\lambda_t}{\delta^t}\right]\left[\frac{\bar{k}u''(\bar{c})}{u'(\bar{c})}\right] - \frac{\lambda_t}{\delta^t}\left[1 - \frac{\bar{c}u''(\bar{c})}{u'(\bar{c})}\right] =$$

$$\mu_0\left[\delta F'(\bar{k}) - 1\right]\left[\frac{\bar{k}u''(\bar{c})}{u'(\bar{c})}\right] - \frac{\lambda_t}{\delta^t}\left[1 - \frac{\bar{c}u''(\bar{c})}{u'(\bar{c})}\right] \tag{66}$$

Consider now the implication of imposing full stationarity, i.e., $(\lambda_t/\delta^t) - (\lambda_{t+1}/\delta^{t+1}) = 0$. Then (63) gives

$$F'(\bar{k}) = \frac{1}{\delta} = 1 + \rho \tag{67}$$

which determines $\bar{k}$, and the right-hand side of (64) becomes

$$RHS = -\lambda_0\left[1 - \frac{\bar{c}u''(\bar{c})}{u'(\bar{c})}\right] \tag{68}$$

Equation (64) becomes

$$u'(\rho\bar{k}) - \gamma v'[F(\bar{k}) - \bar{k}(1+\rho)] = \theta\left[\rho\bar{k}u''(\rho\bar{k}) + u(\rho\bar{k})\right] - \lambda_0\left[1 - \frac{\rho\bar{k}u''(\rho\bar{k})}{u'(\rho\bar{k})}\right] \tag{69}$$

However, if the function $u(c)$ is logarithmic, then the right-hand side of (69) becomes zero, and thus

$$u'(\rho\bar{k}) - \gamma v'[F(\bar{k}) - \bar{k}(1+\rho)] = 0 \tag{70}$$

But generically $\bar{k}$ cannot satisfy both (67) and (70). It follows that in the case where $u(c)$ is logarithmic, a full stationary equilibrium does not exist, and we can only have a semi-stationary equilibrium, with capital stock $\widehat{k}$ that satisfies the following condition, obtained from (64), (65) and (66):

$$u'(\rho\bar{k}) - \gamma v'[F(\bar{k}) - \bar{k}(1+\rho)] = -\mu_0 \left[\delta F'(\bar{k}) - 1\right]/\rho =$$

$$\frac{\delta}{\rho}\left[1 + \rho - F'(\bar{k})\right]\gamma v'[f(\bar{k}) - \bar{k}(1+\rho)] \tag{71}$$

From (71) $\rho > F'(\bar{k}) - 1$ if and only if $u'(\rho\bar{k})$ exceeds $\gamma v'[F(\bar{k}) - \bar{k}(1+\rho)]$. This result is analogous to the one reported at the end of section 2.

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