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Conditional Quantiles of Volatility in Equity Index and Foreign Exchange Data*

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Résumé / *Abstract*

Nous utilisons des techniques d'estimation de modèle reliées à ceux de Galbraith et Zinde-Walsh (2000) pour les modèles ARCH et GARCH, basées sur la *realized volatility* (Andersen et Bollerslev 1998, et autres), afin d'obtenir les quantiles conditionnels de volatilité quotidienne dans les données provenant des marchés boursiers et des marchés de devises étrangères. Ces méthodes nous permettent en principe de caractériser la distribution entière de volatilité en utilisant la volatilité réalisée et les retours carrés. Nous prenons des échantillons de rendements quotidiens et intrajournaliers de l'indice 35 du TSE, et des taux de change DM/$ US et Yen/$ US. Nos résultats montrent également que les percentiles inférieurs de la distribution conditionnelle augmentent proportionnellement moins en périodes de volatilité extrême que les percentiles supérieurs.

This paper uses estimation techniques related to those of Galbraith and Zinde-Walsh (2000) for ARCH and GARCH models, based on realized volatility (Andersen and Bollerslev 1998, and others), to estimate the conditional quantiles of daily volatility in samples of equity index and foreign exchange data. These techniques in principle allow us to characterize the entire conditional distribution of volatility, conditioning on past realized volatility and past squared returns. We take samples of daily and intra-day returns on the Toronto Stock Exchange 35 index, the DM/$ US exchange rate and the Yen/$ US exchange rate. In addition to information about the conditional extremes of volatility, we find some evidence that lower percentiles of the conditional distribution rise proportionately less in high-volatility periods than do the higher percentiles.

Mots clés : Modèle GARCH, volatilité intégrée, régression quantile.

Keywords *: GARCH model, integrated volatility, quantile regression*

JEL Classification : C22, G10

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1. Introduction

Since the observation of Andersen and Bollerslev (1998) that the forecasts of conditional volatility models can be better evaluated by comparison with an estimated volatility based on intra-day quadratic variation than by comparison with squared returns, this concept of *integrated volatility* or *realized volatility* has proven to have numerous applications. Using high-frequency intra-day data to allow computation of these estimates at the daily frequency, a number of authors (in particular Andersen et al. 2001, Andersen et al. 2001 a) have characterized the realized volatility patterns observed in financial time series. Others (Andersen et al. 2001b, Maheu and McCurdy 2000, Galbraith and Zinde-Walsh 2000) have used the integrated volatility in time series estimates of models of volatility, attempting to increase the information available relative to standard GARCH-class models which treat the volatility as a latent variable.

The present paper uses this realized volatility information to produce conditional quantile estimates of models of realized volatility. Quantile regression models, introduced by Koenker and Bassett (1978), allow the investigator to characterize any part of the distribution (given a sufficient sample) of a variable conditional on observable regressors. The technique has enjoyed increasing use, particularly in cross-sectional studies, as computational methods have developed; see for example Buchinsky (1998) for a survey. In the present context, quantile regression allows us to examine (e.g.) the 80th percentile of volatility at time t, conditional on information available up to time $t-1$ on squared returns and integrated volatility.

To do so we use the estimator of Galbraith and Zinde-Walsh (2000) for time series models of volatility using integrated volatility information. This estimator, because it is based on LAD estimation of a representation of the volatility process–that is, on estimation of the conditional median, or 50th percentile–lends itself to estimation of other conditional quantiles. Conditional quantile estimates may be based on any of various models of the conditional volatility; we discuss ARCH and GARCH representations. It is important that the consistency of the estimator does not depend on the high-frequency sampling interval converging on zero in order to provide an arbitrarily accurate estimate of realized volatility; instead the estimation technique used here is robust to measurement error in the daily volatility estimates. This robustness of the technique to imperfect measurement is an important practical advantage because of the well-known imperfections in diffusion approximations to asset prices at very high frequencies, induced by phenomena such as bid-ask bounce.

We are able to characterize the conditional quantiles of an equity price index from the Toronto Stock Exchange, and of the DM/US$ and Yen/US$ exchange rates, and examine the relative impacts of volatile periods on different parts of the distribution. Section 2 describes the techniques that we will use, and section 3 describes the data, filtering and resulting sequences of integrated volatility measures. Section 4 describes the conditional quantiles of volatility, and suggests some differential impacts of high volatility across different parts of the distribution. Consistency and asymptotic normality of the conditional quantile estimator proposed here are examined in the Appendix.

2. Conditional quantile methods

2.1 Models and estimation

In this section we introduce notation and state the assumptions on the data-generating process (DGP) that justify the use of the methods developed in this paper.

Let $\{X_t\}$, $X_t \in \mathcal{R}^n$, be a special martingale process as defined in Andersen et al. (2001). This means that $\{X_t\}$ is a stochastic process adapted to an increasing, right-continuous filtration of σ- fields $\mathcal{F}_t$, $t \in [0, T]$,complete with respect to the corresponding probability measure. The return of the process $\{X_t\}$ over the horizon t, denoted $r(t)$, allows for the unique decomposition:

$$r(t) \equiv X_t - X_0 = M(t) + A(t) \tag{2.1.1}$$

where $M(0) = A(0) = 0$, $M(t)$ is a local martingale and $A(t)$ is locally integrable, of finite variation, and predictable with respect to a filtration $\mathcal{F}_t$. Each component of $\{X_t\}$ is assumed to be a cadlag process. This formulation is very general and includes Itô, jump and jump-diffusion processes.

The process X is sampled at an interval $\frac{1}{h}$, and we will denote the index at this high frequency as τ. We are interested in studying the properties of volatility at a lower frequency with a sampling interval that we define to be equal to one. Throughout this paper, this will correspond to the daily frequency, while the higher frequency will correspond to a number of intra-day observations which varies with the data set. The low- frequency index will be denoted by $t = 1, \ldots, T$; that is, there are h high-frequency observations per low-frequency interval. We define the high-frequency returns $r_h(\tau) \equiv X(\tau) - X\left(\tau - \frac{1}{h}\right)$.

Two questions have to be considered. First, we need to know what requirements the DGP must meet in order for the sums of squared returns and cross-products of returns, or the (empirical) realized covariance matrix $\hat{\sigma}_t^2$, $t \in T$, $\hat{\sigma}_t^2 \in \mathcal{R}^{n \times n}$, defined by

$$\hat{\sigma}_h^2(t) = \sum_{\tau=(t-1)h+1}^{th} r_h(\tau) r_h'(\tau) \tag{2.1.2}$$

to be a valid measure of some latent covariation process. Second, what is the class of DGP's for which the quantile regression models that we will suggest produce consistent estimates of the actual conditional quantiles of volatility of processes in this class?

The answer to the first question was given in a quite general form in Andersen (2001). The corollary of their results relevant to the present study is that for any two scalar semi-martingale processes $\{Y_t\}$ and $\{Z_t\}$, the quadratic variation and covariation processes, defined as

$$\begin{aligned} [Y, Y] &= YY - 2\int Y_- dY \\ [Y, Z] &= YZ - \int Y_- dZ - \int Z_- dY \end{aligned} \tag{2.1.3}$$

can be represented as

$$\begin{aligned} [Y, Y]_t &= [Y^c, Y^c]_t + \sum_{0 \le s \le t} (\Delta Y(s))(\Delta Y(s)) \\ [Y, Z]_t &= [Y^c, Z^c]_t + \sum_{0 \le s \le t} (\Delta Y(s))(\Delta Z(s)), \end{aligned} \tag{2.1.4}$$

where $\{Y\}$ and $\{Z\}$ have been decomposed into a continuous term with infinite variation path and a term representing the compensated jump part of the process[1]. If we denote by $[X, X]_t$ the matrix of quadratic variation and covariation processes of the components of $\{X_t\}$, then the following result holds:

$$\sigma^2(t) \equiv p \lim_{h \to \infty} \hat{\sigma}_h^2(t) = [X, X]_t. \tag{2.1.5}$$

[1]The semi-martingale formulation encompasses all processes conformable with the arbitrage-free assumption. However, it rules out the fractional Brownian motion (BM), $B_d(t)$, $0 < d < \frac{1}{2}$. The fractional BM is not a semi-martingale and allows for arbitrage.

It is important to recognize that $\sigma_h^2(t)$ is generally not measurable with respect to the filtration at the moment $t - \Delta t$. An Itô process with constant volatility, as in the Black-Scholes model, is an exception to this rule. However, the realized volatility measures provide unbiased estimates of the ex-ante conditional variances, in general. The error terms incorporate both the errors of measurement of the realized volatility and innovations orthogonal to the conditioning set.

If it is assumed that the return process $r(t)$ is an Itô process, i.e. that it satisfies the stochastic differential equation

$$r(t) = \int_0^t \mu(s)\, ds + \int_0^t \Omega(s)\, dW(s), \tag{2.1.6}$$

where $W(t) \in \mathcal{R}^n$ denotes a vector of independent Wiener processes, then the *integrated covariance* process, (in the scalar case, the *integrated volatility* process central to Hull-White option pricing) is defined as

$$[X, X]_t = \int_0^t \Omega'(s)\, \Omega(s)\, ds. \tag{2.1.7}$$

The sums of squared high frequency returns and cross-products of high-frequency returns can therefore be used as a measure of this integrated covariance process. Andersen et al. also describe the specification of the jump and mixed jump-diffusion varieties of the special semi-martingale process.

Maheu and McCurdy (2001) suggested that a sum of squared returns and cross-products of returns at high frequency could be used as a measure of realized covariance over the low-frequency period, the realized covariance being defined as the sum of covariances of high-frequency innovations over the period. Consider a discrete return process defined as

$$r_{(h)}(t) = \sigma_{(h)}^2(t)\, z(t), \; z \sim iid(0, I), \; t = \frac{k}{h}, \; k \in \mathcal{N}. \tag{2.1.8}$$

The sum of squared returns will provide an unbiased estimate of the realized covariance over the corresponding time interval.[2]

If the DGP is the semi-martingale process described in Andersen et al. (2001), and if it were possible to increase the sampling frequency without bound $(h \to \infty)$,

[2] It may be possible to unify the discrete time and continuous time notation here by considering cadlag functions that change only on some discrete index.

then the realized volatility would provide an error-free ex-post measure of latent volatility. In practice, the sampling frequency cannot be increased indefinitely; it is typically impractical to increase sampling frequency beyond a certain point because of errors induced by market microstructure (for example, bid/ask bounce). These considerations lead to the recognition that the realized volatility is estimated by the sum of squared returns with an error, that is,

$$\sigma^2(t) = \hat{\sigma}_h^2(t) + \varepsilon_t. \tag{2.1.9}$$

Of course, in the case of the discrete DGP error-free measurement of the realized volatility is an impossibility even in theory.

Consider now the question of the class of DGP's for which we can obtain conditional quantile estimates using the realized volatility.[3] We will use an estimator which projects the realized volatilities onto past squared returns, an analogue of the AR-approximation estimator used for ARMA processes by Galbraith and Zinde-Walsh (1997) and, more directly, of the GARCH model estimator of Galbraith and Zinde-Walsh (2000). The estimates are shown in the Appendix to be consistent and asymptotically normal in spite of the presence of measurement noise in the estimates of daily volatility, which as we have noted is an important feature given the well-known imperfection of the diffusion approximation at extremely high frequencies.

We now suppress the explicit dependence of realized volatility measures on h, and use a subscript t to denote the lower-frequency (daily) index. The GARCH(p,q) model $\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2$, together with (2.1.9), yields the model with realized volatilities:

$$\hat{\sigma}_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \hat{\sigma}_{t-i}^2 - \sum_{i=1}^{p} \beta_i e_{t-i} + e_t. \tag{2.1.10}$$

Following Galbraith and Zinde-Walsh (2000), we avoid the difficulty arising from the MA(p) error in (2.1.10) by considering the representation

$$\sigma_t^2 = \kappa + \sum_{\ell=1}^{\infty} \nu_\ell \varepsilon_{t-\ell}^2, \tag{2.1.11}$$

[3]Because the additional information in realized volatility is used, these estimates are distinct from the class of ARCH conditional quantile estimates considered by Koenker and Zhao (1996).

which is based on the ARCH(∞) representation of the GARCH model. The class of processes for which such an ARCH(∞) representation exists is quite broad; see Giraitis et al. (2000), who give general conditions for its validity for the strong GARCH(p, q) case. Only the existence of the first moment and summability of the coefficients ν_ℓ are required for existence of a strictly stationary ARCH(∞) solution.

Galbraith and Zinde-Walsh (2000) show that we can deduce parameter estimates for a general GARCH(p, q) model (2.1.10) using a truncated version of this ARCH(∞) representation (2.1.11), that is,

$$\hat{\sigma}_t^2 = \kappa + \sum_{\ell=1}^{k} v_\ell \varepsilon_{t-\ell}^2 + e_t. \tag{2.1.12}$$

The truncation parameter k must be such that $k \to \infty$, $k/T \to 0$ for consistent estimation of the GARCH model. These parameter estimates avoid the problem of measurement error in the realized volatility which would arise in estimating the GARCH model (2.1.10) directly using the realized volatility terms as regressors. Given estimates $\{\hat{v}_i\}_{i=1}^k$ of the parameters of (2.1.12), GARCH parameter estimates may be deduced from the deterministic relations between the GARCH parameters and the parameters of the ARCH(∞) representation; see Galbraith and Zinde-Walsh (2000) for the expressions.

From the point of view of the present study, the method of estimation of the ARCH(k) model is crucial. While LS estimates are admissible for some processes, Galbraith and Zinde-Walsh (2000) show that estimation of the initial representation (2.1.11) by LAD is robust to a wider variety of processes $\{\varepsilon_t\}$, including some for which moments may be unbounded.

This LAD estimator, or conditional median, is easily extended to estimation of other conditional quantiles. The LAD criterion function is

$$\min_\beta T^{-1} \sum_{i=1}^{T} |y_i - x_i'\beta|; \tag{2.1.13}$$

the criterion function for the conditional θ−th quantile is (see, e.g., Koenker and Bassett 1978)

$$\min_\beta T^{-1} \left\{ \sum_{y_i \geq x_i'\beta} \theta |y_i - x_i'\beta| + \sum_{y_i < x_i'\beta} (1-\theta)|y_i - x_i'\beta| \right\}, \tag{2.1.14}$$

which returns the LAD minimand ($\times\frac{1}{2}$) for $\theta = 0.5$, the conditional median.

Numerical minimization of this expression can be performed in approximately the time required for computation of LS estimates, for large model orders. The quantile estimates are obtained using the same numerical methods as LAD estimates.

Below we apply (2.1.14) to obtain estimates of the ARCH(k) representation for the conditional quantiles $\theta = \{0.1, 0.2, \ldots, 0.9\}$. From these, estimated historical conditional quantiles and parameters of estimated GARCH representations are obtained for each data set.

2.2 Asymptotic inference

We now obtain the asymptotic distributions of the estimated parameters of the conditional quantile models.

Consider an infinite-dimensional data generation process:

$$y_t = X_t(\infty)\,\gamma(\infty) + e_t \tag{2.2.1}$$

with a summable sequence of parameters $\gamma(\infty)$. We assume also that $x_{0t} = 1$ for $t = 1, 2, ...$, with x_{0t} being the first element of $X_t(\infty)$. Let $F(x)$ denote the distribution function of e_t and let $\chi_q = F^{-1}(q)$, that is, the value corresponding to the $q-$th quantile of this distribution. Note that $F(0) = 0.5$ so that $F^{-1}(0.5) = \chi_{0.5} = 0$.

Condition A1

(a) The marginal distribution of e_t has density $\phi(\chi_q)$ at quantile q, and $\phi(\chi_q) > 0$ and is continuous in some neighbourhood of χ_q.

(b) The sequence $\{e_t\}$ is independently and identically distributed.

For a $k \times \ell$ matrix A, we denote by $\max|A|$ the largest element, that is $\max_{i,j} |\{A\}_{ij}|$.

Condition A2

For an increasing sequence $\{\mathcal{F}_t\}$ of sigma fields and a sequence of possibly random matrices $\{V_T(k)\}$, assume that for every k that $X_t(k)$ is measurable with respect to $\mathcal{F}_t$ and for some monotonically increasing function ω, such that $k = \omega(T)$ as $T \to \infty$,

(a) $\sup_{1 \le t \le T} \max|V_T(k)^{-1} X_t(k)'| = o_p(1)$;

(b) $\max\left|\sum_t V_T(k)^{-1} X_t(k)' X_t(k) V_T(k)^{-1} - I_{k+1}\right| = o_p(1)$;

(c) $\sup_{1\leq t\leq T}|X_t([k+1,\infty))\gamma^{\infty}([k+1,\infty))| = o_p(1)$;
(d) e_t is independent w.r.t. $\mathcal{F}_t$.

Next define $\vec{\chi}_q(\infty) = (\chi_q, 0, 0, ...) \in \Re^{\infty}$ and $\vec{\chi}_q(k) = (\chi_q, 0, ..., 0) \in \Re^{k+1}$.

Theorem 1. *Under the conditions A1 and A2 as $T \to \infty$, $k = \omega(T)$, the asymptotic distribution of any finite $\tilde{k}+1$ - subvector of the quantile q estimator, $(\hat{\gamma}_q(k))_{[0,\tilde{k}]}$, is given by*

$$V_T(k)(\hat{\gamma}_q(k) - \gamma^{\infty}(k) - \vec{\chi}_q(k)) \xrightarrow{D} N\left(0, \left(\frac{q(1-q)}{\phi^2(\chi_q)}\right)I\right), \tag{2.2.2}$$

where I is an identity matrix and convergence is defined for all finite-dimensional distributions.

To apply the results of the theorem in the context of ARCH(∞) and GARCH quantile estimation, the following additional conditions are sufficient:

Condition A3

(a) The squared returns $\{\varepsilon^2\}$ form a strictly stationary ergodic sequence;

(b) The conditional variance is in the class of stationary, invertible semi-strong GARCH models that have the ARCH(∞) representation;

(c) The measurement error e_t is independent of past values of the squared returns.

Condition A4

$E\left[\left(\varepsilon^2\right)^4\right]$ is finite.

The problem of estimating the conditional quantiles of volatility can be rewritten using the notation of the theorem as follows:

$$\hat{\sigma}_t^2 = X_t(k)\gamma(k) + X_t[k+1,\infty)\gamma[k+1,\infty) + e_t,$$

where $X_t(k) \equiv \left(1, \varepsilon_{t-1}^2, ..., \varepsilon_{t-k}^2\right)$ and $\gamma(k) \equiv (\kappa, \nu_1, ..., \nu_k)'$. We truncate the model at lag k :

$$\hat{\sigma}_t^2 = X_t(k)\gamma(k) + u_t \tag{2.2.3}$$

and compute the quantile estimates of (2.2.3).

The matrix V_T is not needed for computing the estimates $\hat{\gamma}_q(k)$ themselves, but we need to define this matrix constructively to perform asymptotic statistical inference on the estimates. Define the symmetric matrix

$$\hat{\Omega}_T(k) = (T-k)^{-1}\sum_{t=k+1}^{T} X_t'(k)\,X_t'(k).$$

By the ergodic theorem, $\hat{\Omega}(k) \to_p \Omega(k) = E[X_t'(k) X_t(k)]$ for any k. We can now define $V_T(k)$ as

$$V_T(k) = (T-k)^{\frac{1}{2}} \Omega^{\frac{1}{2}}(k); \qquad (2.2.4)$$

$\hat{V}_T(k)$ is defined analogously, using $\hat{\Omega}_T(k)$.

Another necessary element for the purposes of inference is the density of the innovation e_t at the quantile q, $\phi(\chi_q)$. To estimate this, compute the residuals that correspond to the vector of coefficients $\hat{\gamma}_q(k)$ estimated at the quantile q:

$$\hat{e}_{qt} = \hat{\sigma}_t^2 - X_t(k)\hat{\gamma}_q(k).$$

Let $\hat{\psi}_q(\cdot)$ denote the kernel estimator of the density of $\{\hat{e}_{qt}\}$. We use the quantity $\hat{\psi}_q(0)$ as the estimator of $\phi(\chi_q)$ (note that if the model is correctly specified, then $E\left[\hat{\psi}_q(x-\chi_q)\right] = \phi(x)$ for x in the support of ϕ and for any quantile q such that $\phi(\chi_q) \neq 0$). With this estimate of $\phi(\chi_q)$ and with $\hat{V}_T(k)$ substituted for $V_T(k)$, the estimate of the variance of $\hat{\gamma}(k)$ is obtained from (2.2.2).[4]

3. Data and estimates

3.1 Data

As we have noted, two types of data are used in this study: an equity-price index and a set of foreign exchange prices. The former is a short (one-year) span of very high-frequency data, spaced fifteen seconds apart, on the Toronto Stock Exchange index of thirty-five large-capitalization stocks (TSE 35), for calendar year 1998.[5] The latter is a fourteen-year sequence of observations spaced at five minutes, and pertain to the Deutschmark / \$US and Yen /\$US exchange rates.

The fifteen-second intra-day data on the TSE 35 index value (as well as bid and ask) are available through the 9:30 a.m. to 4:00 p.m. trading day, for a total of approximately 1560 observations per day. The data must be filtered to recognize the facts that trading does not take place throughout the 24-hour day, and that there are occasional anomalies near the beginning of the trading day. First, we treat the change in the index value between the 4:30 close and 9:30 open on the following day

[4]The covariance matrix of estimated GARCH parameters can be obtained from the Jacobian of the transformation from ARCH to GARCH parameters, as $J'(\text{var}\gamma(\hat{k}))J$, where J is the Jacobian.

[5]This index has since been superseded by the S & P/TSE 60 Index of large-capitalization stocks.

as a contribution to volatility (which if ignored would lead to underestimation of $\hat{\sigma}_t^2$). Second, the first few minutes of the trading day typically show the index value outside the bid/ask range; within the first two minutes of trading, the index value is usually again within the range. We therefore use the midpoint between bid and ask for the first two minutes of the trading day, by which point the two measures are almost invariably compatible.

Finally, of course, we must decide on a level of time aggregation, or h in the notation of section 2. Since the raw data are provided at 15-second intervals, summing four squared returns to obtain a single intra-day observation, that is aggregating to one-minute returns, implies $h = 1560/4 = 390$ observations per day; aggregation to the five-minute interval, corresponds to $h = 1560/20 = 78$; we present results for both of these aggregations. Each of the filtering operations applied to these TSE 35 data are described more fully in Galbraith and Kisinbay (2000).

The foreign exchange data used in this study are taken from the HFDF 2000 data set compiled and distributed by Olsen Group, Switzerland. Foreign exchange returns recorded every five minutes span the period from January 2, 1986, 00:00:00 GMT to January 1, 1999, 23:35:00 The returns are computed as the mid-quote price difference, expressed in basis points (i.e. multiplied by 10,000). The mid-quote price at the regular time point is estimated through a linear interpolation between the previous and following mid-price of the irregularly spaced tick-by-tick data. The average bid-ask spread over the last 5-minute interval is expressed in basis points. If there is no quote during this interval, the mean bid-ask spread is zero. Currencies are traded continuously throughout the day, seven days a week; thus the data set contains 1,262,016 5-minute returns, expressed in USD terms.

In order to compute integrated volatilities, it is necessary to perform some filtering. We have followed Bollerslev and Domowitz (1993) and other researchers, in defining the trading day t as the interval from 21:05GMT of the previous calendar day to 21:00GMT on the calendar day t. The estimate of daily realized volatility, that is, the daily integrated volatility, is computed by summing squares of currency's five-minutes returns over the day.

Although currencies are traded continuously, there are periods such as weekends and holidays, during which trading activity is very low. Following Andersen, Bollerslev, Diebold and Labys (1999) and other authors, we have filtered such low-trading-volume days out of the data.[6] The filters that we applied to the data also

[6] An alternative to eliminating weekends would be to apply a corresponding seasonal

eliminated weekends, fixed holidays (December 24-26, 31, January 1-2) as well as moving holidays (Good Friday, Easter Monday, Memorial Day, the Fourth of July, Labour Day, Thanksgiving (US) and the day after Thanksgiving. In addition, we have eliminated from the data the days for which the indicator variable (the bid-ask spread) had 144 or more zeroes, thus corresponding with the technical "holes" in the recorded data. Application of all these filters reduced the data sets to 876,096 data points, or 3042 days, for the DM/\$US, and 877,248 or 3046 days for the Yen/\$US.

3.2 ARCH(k) conditional quantile estimates

Before considering estimates of parameters of GARCH (or other, such as FI-GARCH) representations of conditional volatility, we use the ARCH(k) representations directly to characterize the historical conditional quantiles of volatility of our data series, and to consider the relative impacts of high-volatility episodes on different parts of the volatility distribution.

Each of Figures 1(a–d), 2(a–d) and 3(a–d) pertains to one of the sets of high-frequency returns, respectively the TSE 35 index returns (252 daily observations), the DM/\$US returns and Yen/\$US returns (3042 and 3046 daily observations respectively). The results plotted in the figures correspond to a number of intra-day observations h of 78 (TSE 35) and 288 (each set of foreign exchange returns). Figure ia, $i = 1, 2, 3$, plots the realized volatility sequence over each of the sample periods. These sequences are much smoother that the corresponding sequences of daily squared returns (not shown).

Figure ib depicts the sequence of conditional 20th, 50th and 80th percentiles of volatility, conditioning on the set of past squared returns used by the ARCH(k) representation; $k = 8$ on the first data set and $k = 12$ on the larger sets of foreign exchange data. In the rest of this sub-section, we consider a few noteworthy results emerging from these estimated conditional quantiles. In later sections we consider the parameters of GARCH-type (linear) conditional quantile models.

First, consider Figure ib and the conditional 80th and 20th percentiles of volatility relative to the conditional median. Since the volatility is skewed to the right, the distance between 80th and 50th percentiles of conditional volatility almost invariably exceed the distance between 50th and 20th. The ratio of 80th to 50th percentiles typically exceeds the ratio of 50th to 20th as well.[7]

filter to the data.

[7]These features are not necessarily clear from graphical display of the results, but are clear in numerical displays of these quantities.

Next consider the impact of high-volatility periods on these conditional quantiles, and ask whether either the upper or lower quantiles tend to experience greater proportionate change in such periods. Figures ic and id plot the ratios of 80th to 20th and 90th to 10th conditional percentiles of conditional volatility . We ask whether this ratio tends to be systematically higher, or lower, in relatively high-volatility periods.

Visual inspection of ic and id suggests some positive association, particularly in the foreign exchange data (where the larger sample size allows more precise estimation of the relatively distant quantiles). Such an association–that is, upper percentiles of volatility show a proportionately greater movement in high-volatility periods–would suggest that using the change in conditional median (similarly, conditional mean) volatility as an indicator of changes in the upper percentiles of volatility would lead to underestimation of the probability of extremely high realizations of daily volatility.

Table 1 quantifies the evidence on this association by recording the correlations between sequences of realized volatilities (Figures ia) and the ratios of upper- to lower- quantiles (Figures ic and id).

Table 1

Correlation of realized volatility with ratios of upper- to lower conditional quantiles[8]

Data set	T	corr(rv, $\frac{80th}{20th}$)	corr(rv, $\frac{90th}{10th}$)
TSE 35	252	0.199	0.028
		(0.063)	(0.064)
		(0.122)	(0.043)
DM/$US	3042	0.288	0.326
		(0.017)	(0.017)
		(0.030)	(0.031)
Yen/$US	3046	0.106	0.286
		(0.018)	(0.017)
		(0.065)	(0.083)

[8]The first bracketed quantity below each measured correlation is the conventional standard error; the second is the Newey-West HAC standard error with lag length of 12; these standard errors are little affected by small changes in p.

These correlations suggest a substantial positive association of the type just described; the associated standard errors (conventional and heteroskedasticity & autocorrelation consistent) are comparatively small in the two larger data sets, implying correlations which are significantly greater than zero at conventional test levels in three of the four cases (the exception being the 28/20 correlation in Yen/$US data, which is within two standard errors of zero); in the much smaller TSE data set, the correlations are not significant by the Newey-West standard errors. Such a relation implies that characterizing changes in conditional volatility by the conditional mean or median alone will not provide an undistorted indicator of changes in the conditional upper quantiles. The positive associations which we appear to observe here imply that the probabilities of extremely high realizations of volatility are higher in high-volatility periods than would be the case if all conditional quantiles increased proportionately in such periods.

In the next section, we examine the parameter estimates of models of the conditional quantiles of volatility.

4. Estimated models of conditional quantiles of volatility

We now consider parameter estimates of GARCH models of the conditional quantiles, estimated using the methods given in section 2. We begin with the standard GARCH(1,1) model, and then consider alternative GARCH models which may provide better fits for some quantiles.[9]

Table 2 provides estimates of the parameters in the foreign exchange return series over the quantiles 0.1 to 0.9.

[9]Throughout this section the number of approximating ARCH terms used in the GARCH model estimation is $k = 14$ for the foreign exchange data and $k = 8$ for the smaller equity index data set. These orders are approximately those implied by the rule of thumb $k = 8 + int(2\ln(T/100))$ stated in Galbraith and Zinde-Walsh (1997) in an autoregressive context.

Table 2
GARCH(1,1) models of conditional quantiles
Foreign exchange returns

	DM/$US			Yen/$US		
Quantile	ω	β	α	ω	β	α
0.1	1.0×10^{-5}	0.607	0.059	0.064	0.594	0.065
	(9.0×10^{-7})	(0.028)	(0.003)	(0.007)	(0.032)	(0.003)
0.2	1.2×10^{-5}	0.595	0.079	0.072	0.632	0.084
	(1.1×10^{-6})	(0.025)	(0.003)	(0.007)	(0.024)	(0.003)
0.3	1.2×10^{-5}	0.640	0.087	0.091	0.596	0.108
	(1.1×10^{-6})	(0.023)	(0.003)	(0.007)	(0.020)	(0.004)
0.4	1.5×10^{-5}	0.634	0.100	0.102	0.609	0.122
	(1.3×10^{-6})	(0.024)	(0.004)	(0.008)	(0.019)	(0.004)
0.5	1.5×10^{-5}	0.655	0.112	0.120	0.590	0.141
	(1.3×10^{-6})	(0.023)	(0.004)	(0.008)	(0.017)	(0.004)
0.6	1.6×10^{-5}	0.689	0.118	0.130	0.613	0.154
	(1.5×10^{-6})	(0.023)	(0.004)	(0.008)	(0.016)	(0.004)
0.7	1.7×10^{-5}	0.695	0.134	0.170	0.579	0.183
	(1.8×10^{-6})	(0.024)	(0.005)	(0.010)	(0.017)	(0.004)
0.8	2.0×10^{-5}	0.686	0.171	0.251	0.480	0.253
	(2.3×10^{-6})	(0.027)	(0.007)	(0.016)	(0.019)	(0.006)
0.9	2.5×10^{-5}	0.698	0.194	0.360	0.423	0.403
	(4.1×10^{-6})	(0.038)	(0.013)	(0.032)	(0.026)	(0.012)

As an example of the interpretation of these numbers, consider the DM/$US exchange rate. Conditional on a previous-period realized volatility of 0.0004 and squared return of 0.00015, the 10th and 90th percentiles of the conditional distribution of realized volatility lie at 0.00026 and 0.00033. That is, 10% of realized values of volatility would be below the first number and above the second, given the conditions mentioned. This calculation therefore allows daily computation of the quantiles of the conditional distribution of the next day's volatility, given conditions just observed.

As we consider higher quantiles, we might expect to see higher values of each of the parameters. However, we in fact observe monotonic increases in ω and α only;

β does not display a regular pattern of increase. That is, the higher fitted values of conditional volatility implied by a higher quantile project onto the intercept and the coefficient on the lagged squared return, but do not appear typically to imply a higher weight on the lagged realized volatility. This pattern continues to hold in the higher-order models estimated in Table 4 below.

On the smaller data set of equity returns, we do not estimate the 10th and 90th percentiles, but give in Table 3 the conditional quantiles 0.2, 0.5 and 0.8.

Table 3
GARCH(1,1) models of conditional quantiles[10]
TSE 35 equity index returns

Quantile	ω	β	α
0.2	5×10^{-6}	0.70	0.05
	(4×10^{-6})	(0.14)	(0.02)
0.5	6×10^{-6}	0.76	0.10
	(4×10^{-6})	(0.07)	(0.02)
0.8	2×10^{-5}	0.55	0.16
	(8×10^{-6})	(0.08)	(0.02)

We see in the equity index data the same pattern: higher quantiles of the conditional volatility distribution are reflected in larger values of ω and α.

Next consider the specification of alternative GARCH models for the conditional quantiles. Where the diffusion approximation is imperfect, additional terms in the model may capture useful conditioning information not present in the GARCH(1,1) specification. In Table 4 we report the results of GARCH model selection using the Bayesian Information Criterion (BIC) and the Final Prediction Error (FPE), from the set of model orders $(p, q), p = 1, \dots 4,\ q = 1, \dots 4$. The chosen optimal order is the same for the two criteria in each case.

[10]Standard errors of the estimates are given in parentheses.

Table 4
GARCH model selection by quantile
Foreign exchange returns

	DM/$US			Yen/$US		
Quantile	(p^*, q^*)	$\sum \beta_i$	$\sum \alpha_i$	(p^*, q^*)	$\sum \beta_i$	$\sum \alpha_i$
0.1	(4,1)	0.709	0.059	(4,1)	0.733	0.065
0.2	(4,2)	0.623	0.113	(3,4)	0.825	0.061
0.3	(3,2)	0.700	0.112	(2,2)	0.568	0.114
0.4	(4,1)	0.768	0.101	(2,2)	0.601	0.146
0.5	(4,1)	0.754	0.126	(2,2)	0.472	0.187
0.6	(3,2)	0.778	0.127	(2,2)	0.467	0.227
0.7	(4,3)	0.760	0.165	(2,2)	0.474	0.247
0.8	(3,4)	0.711	0.229	(3,2)	0.550	0.282
0.9	(2,2)	0.687	0.264	(4,1)	0.547	0.403

The information criteria favour models with at least four GARCH parameters in addition to the intercept. In these higher-order models as well we see the larger values associated with higher quantiles arising through larger values of the α's, that is, the terms in lagged squared returns. The intercept values (not recorded in the Table 4) also increase monotonically with the quantile, as in the results of Table 2, while there is no pattern of regular increase in the coefficients on lagged realized volatility: that is, the higher quantiles do show similar persistence of volatility, and the higher values are reflected in higher weight on recent squared returns.

6. Concluding remarks

We have developed an estimator for the conditional quantiles of volatility using the realized volatility measure emphasized by Andersen and Bollerslev (1998). This estimator retains the robustness to measurement noise of the LAD-based estimator of Galbraith and Zinde-Walsh (2 000); consistency and asymptotic normality of the estimator of the conditional quantiles is demonstrated without requiring that the daily volatility estimate become arbitrarily accurate through a high frequency sampling interval converging on zero. This point is particularly important because in practical application, despite the very high frequencies of sampling sometimes available, market micro-structure effects such as bid-ask bounce invalidate the diffusion approximation at very high frequencies. In practice, therefore, one must place some limit on the frequency of sampling, implying that consistency cannot be obtained via complete convergence of the daily volatility estimates to the true values.

Application of the estimator to both foreign exchange and equity index data suggests some regularities not detectable in conditional mean (or median) volatility models. First, there is evidence in each data set that higher conditional quantiles increase more than proportionately in high-volatility periods, underlining the value of a more complete characterization of the conditional distribution of volatility in understanding such periods. Second, also in each data set, we observe a common pattern in the GARCH model coefficients of different conditional quantiles. The pattern is such that the GARCH-type characterizations of higher quantiles tend to show higher values of the intercept parameter and the coefficient on lagged squared returns than for lower quantiles, while the lagged volatility does not show any regular pattern of increase; that is, the higher fitted values of volatility associated with higher quantiles project onto the intercept and lagged squared return in the GARCH-type characterization . A further possibility, to be investigated in future work, is that non-linear models of the conditional quantiles would show some regular pattern of shape change that is not captured in the GARCH class of model.

Appendix

Proof of Theorem 1. Define the check function (a generalization of the absolute value function):

$$f_q(x) = \left[\left(q - \frac{1}{2} \right) + \frac{1}{2} sgn(x) \right] x.$$

Using this check function we can now define

$$\begin{aligned} &Z_T(\gamma, k; q) = \\ &\quad \sum f_q\left(e_t - X_t(k)\left(\gamma(k) - \gamma^{\infty}(k) - \vec{\chi}_q\right) + X_t[k+1,\infty)\gamma^{\infty}[k+1,\infty)\right) \\ &\qquad - \sum f_q\left(e_t - \chi_q + X_t[k+1,\infty)\gamma^{\infty}[k+1)\right). \end{aligned}$$

Note that $Z_T(\gamma, k; q)$ is minimized by the quantile estimator. Following Phillips (1995), who considered generalized functions for LAD asymptotics, we choose a set of test functions Ψ such that $\psi(x) \in \Psi$. We assume that functions in Ψ are twice continuously differentiable and have a compact support, and normalize these functions such that $\int \psi(x)\, dx = 1$. Using this set of test functions, a sequence of smoothed functions, defined by

$$f_q^m(x) = m \int f_q(x - v)\, \psi(mv)\, dv,$$

converges weakly to the check function, treated as a generalized function (similarly to Phillips 1995, Galbraith and Zinde-Walsh 2000).

Now define $g(k) = V_T\left(\gamma(k) - \gamma^{\infty}(k) - \vec{\chi}_q(k)\right)$, and the smoothed process $Z_T^m(\gamma, k; q)$:

$$\begin{aligned} &Z_T^m(\gamma, k; q) = \\ &\quad \sum f_q^m\left(e_t - \chi_q - X_t(k)\, V_T^{-1} g(k) + X_t[k+1,\infty)\gamma^{\infty}[k+1,\infty)\right) \\ &\qquad - \sum f_q^m\left(e_t - \chi_q + X_t[k+1,\infty)\gamma^{\infty}[k+1)\right). \end{aligned}$$

Consider the Taylor expansion of $Z_T^m(\gamma, k; q)$ around $g = 0$:

$$\begin{aligned} &Z_T^m(\gamma, k; q) = \\ &\quad - \sum f_q^{m\,\prime}\left(e_t - \chi_q + X_t[k+1,\infty)\gamma^{\infty}[k+1,\infty)\right) X_t(k)\, V_T^{-1}(k)\, g(k) \\ &\quad + \frac{1}{2} \sum f_q^{m\,\prime\prime}\left(e_t^* - \chi_q + X_t[k+1,\infty)\gamma^{\infty}[k+1,\infty)\right) \\ &\quad \times g'(k)\, V_T^{-1} X_t(k)'\, X_t(k)\, V_T^{-1} g(k), \end{aligned} \qquad (A1)$$

where $e_t^* \in \left(e_t, e_t - X_t(k) V_T^{-1}(k) g(k)\right)$.

Note that, as a generalized function, $f_q'(x) = (q - \frac{1}{2}) + \frac{1}{2} sgn(x)$ and $f_q''(x) = \delta(x)$, where $\delta(x)$ denotes the Dirac delta function.

We will establish asymptotic limits of the expansion (A1), and will show that

$$\sum \left[\frac{1}{\sigma_m}\left(f_q^{m\prime}(e_t - \chi_q + X_t[k+1,\infty)\gamma^\infty[k+1,\infty)) - \mu_q^m\right)\right] X_t(k) V_T^{-1}(k) \qquad (A2)$$
$$\xrightarrow{D} N(0, I),$$

where
$\mu_q^m = E\left[f_q^{m\prime}(e_t)\right] = E\left[f_q^{m\prime}(e_t) I\left(|e_t| < m^{-1}\right)\right]$ and $(\sigma_m^q)^2 = E\left[\left(f_q^{m\prime} - \mu_q^m\right)^2\right]$.
As well,

$$\sup_{\|g(k)\| < C_g} \sum \left\{B - E\left[f_q^{m\prime\prime}(e_t)\right] I_{k+1}\right\} \xrightarrow{p} 0, \qquad (A3)$$

where $B =$

$$\left[f_q^{m\prime\prime}(e_t^* - \chi_q + X_t[k+1,\infty)\gamma^\infty[k+1,\infty)) \times V_T(k)^{-1} X_t(k)' X_t(k) V_T(k)^{-1}\right],$$

and where I_k is the $k \times k$ identity matrix, for any constant $C_g > 0$.

To prove (A2) note that from A1(c),
$\sup\left|f_q^{m\prime}(e_t - \chi_q + X_t[k+1,\infty)\gamma^\infty[k+1,\infty)) - f_q^m(e_t - \chi_q)\right| = o_p(1)$, so that we can ignore $X_t[k+1,\infty)\gamma^\infty[k+1,\infty)$ in the argument of $f_q^{m\prime}$ in (A3). Define

$$\xi_{T,t} = \left(\frac{1}{\sigma_m^q}\right)\left[f_q^{m\prime}(e_t - \chi_q) - \mu_q^m\right] X_t(k) V_T^{-1}(k) \lambda$$

for some $(k+1) \times 1$ vector λ such that $\lambda'\lambda = 1$, and for an arbitrary small $\varepsilon > 0$, define

$$\zeta_{T,t} = \xi_{T,t} I\left(\sup_{1 \le t \le T} \max\left|V_T(k)^{-1} X_t(k) < \varepsilon\right|\right).$$

Since $f_q^{m\prime}$ is a bounded function and given A2, we have that $|\sum \xi_{T,t} - \sum \zeta_{T,t}| \xrightarrow{p} 0$.

We now show that the martingale difference array $\{\zeta_{T,t}\}$ satisfies the conditions of the central limit theorem of McLeish (see, e.g., Bierens 1994, Theorem 6.1.6). Indeed, $\sup_{T \ge 1} E\left[\max_t(\zeta_{T,t})\right] < \varepsilon^2 E\left[\left(f_q^{m\prime} - \mu_q^m\right)^2\right] < \infty$ and the condition (a) of the McLeish theorem is satisfied. The condition (b) follows from (A2) (a) and the fact that $f_q^{m\prime}$ is bounded and $\sigma_m^2 > 0$. For condition (c) of the theorem we need to show that $\sum_{t=1}^T \zeta_{T,t}^2 \to^p 1$. Define

$$\eta_{T,t} = \zeta_{T,t}^2 - \lambda' V_T(k)^{-1} X_t(k)' X_t(k) V_T(k)^{-1} \lambda$$
$$\times I\left(\sup_{1 \le t \le T} \max\left|V_T(k)^{-1} X_t(k) < \varepsilon\right|\right).$$

By the independence of $\{e_t\}$ the η_t are uncorrelated, the LLN applies providing that $\sum \eta_t \xrightarrow{p} 0$, and so $\lim_{T\to\infty}\left(\sum_t \xi_t^2\right) = 1$.Thus, by the CLT $\sum_t \xi_{T,t} \xrightarrow{D} N(0,1)$. We can obtain (A3) similarly by using A2 (b) and continuity and boundedness of $f_q^{m\prime\prime}(x)$.

The minimizer $g_T^m(k)$ of $Z_T^m(g,k;q)$ satisfies the first-order conditions:

$$\begin{aligned} 0 = &-\sum f_q^{m\,\prime}\left(e_t + X_t[k+1,\infty)\gamma^{\infty}[k+1,\infty)\right) X_t(k)\, V_T^{-1}(k) \\ &+ \frac{1}{2}\sum f_q^{m\,\prime\prime}\left(e_t^* + X_t[k+1,\infty)\gamma^{\infty}[k+1,\infty)\right) V_T^{-1} X_t(k)' X_t(k)\, V_T^{-1} g_T^m(k). \end{aligned}$$

Consider the limiting process $Z(\gamma;q)$ to which the finite dimensional distributions of $Z_T^m(\gamma,k;q)$ converge as $T\to\infty$. By the convexity of $Z_T^m(\gamma,k;q)$ it follows (see Phillips, 1995) that this limit process is uniquely minimized at the limit g of the minimizers of $Z_T^m(\gamma,k;q)$.

Direct computation shows that $\lim_{m\to\infty}\mu_q^m = 0$, $\lim_{m\to\infty}\left(\sigma_q^m\right)^2 = q(1-q)$, and $\lim_{m\to\infty}\mathrm{E}\left[f_q^m(e_t-\chi_q)\right] = \phi(\chi_q)$. Thus we can conclude that

$$g(k) \sim N\left(0, \frac{q(1-q)}{\phi^2(\chi_q)} I_{k+1}\right).$$

Following the argument of Phillips, the terms of the decomposition of $Z_T^m(\gamma,k;q)$, as $m\to\infty$, represent generalized functions that are correspondent terms in $Z_T(\gamma,k;q)$, and the limits of those terms as $T\to\infty$ are ordinary functions (see (A2) and (A3)). We may conclude that the limit process for $Z_T(\gamma,k;q)$ is also $Z(\gamma;q)$ and the minimizer of $Z_T(\gamma,k;q)$ has the same limit process as $g(k)$.

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Figure 1a: TSE 35 Index returns
Realized volatility, five-minute returns

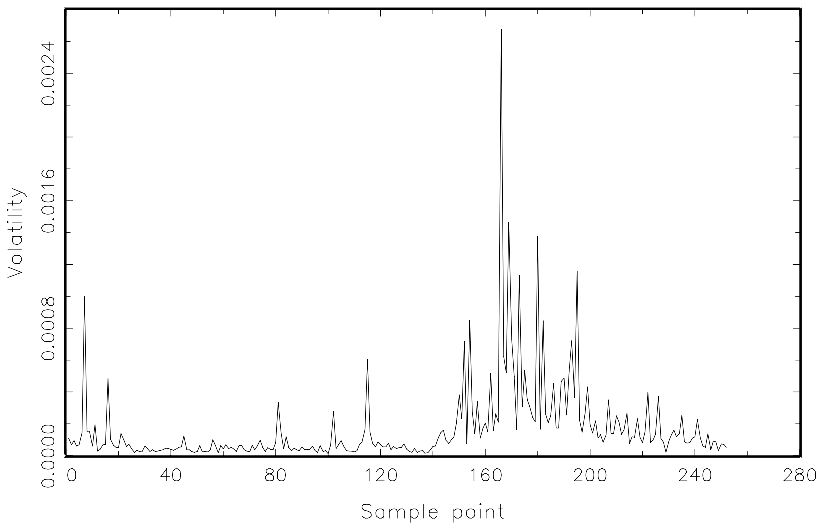


Figure 1b: TSE 35 Index returns
Conditional 20th, 50th, 80th percentiles of volatility
ARCH representation

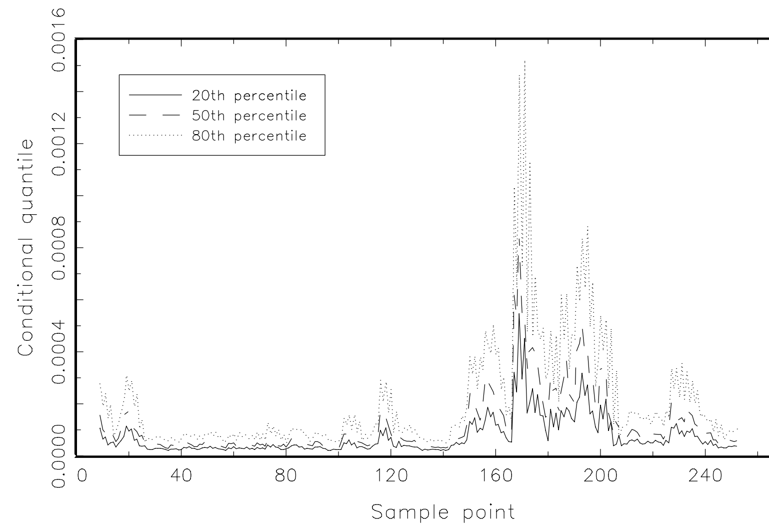


Figure 1c
Ratio of conditional 80th percentile to conditional 20th percentile
ARCH representation

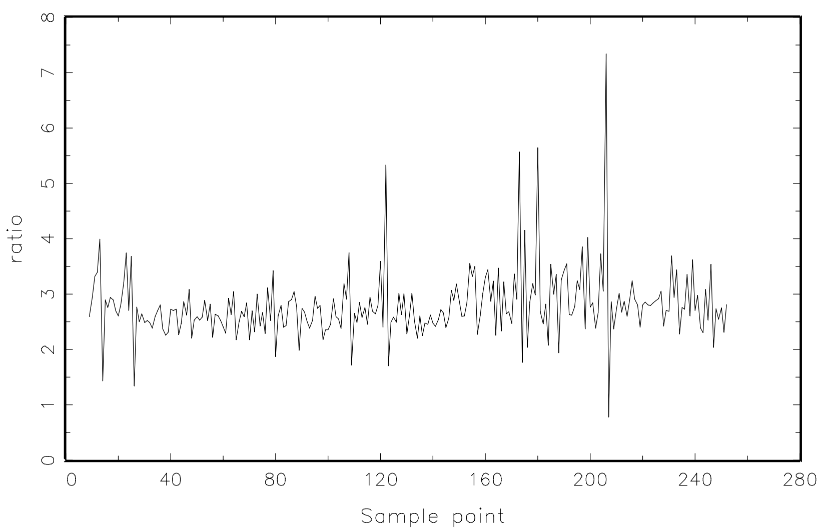


Figure 1d
Ratio of conditional 90th percentile to conditional 10th perce
ARCH representation

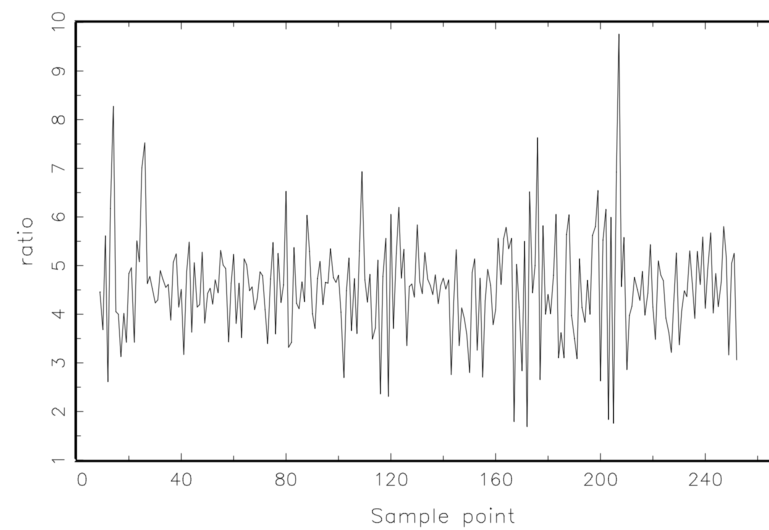


Figure 2a: DM/USD returns
Realized volatility, five-minute returns

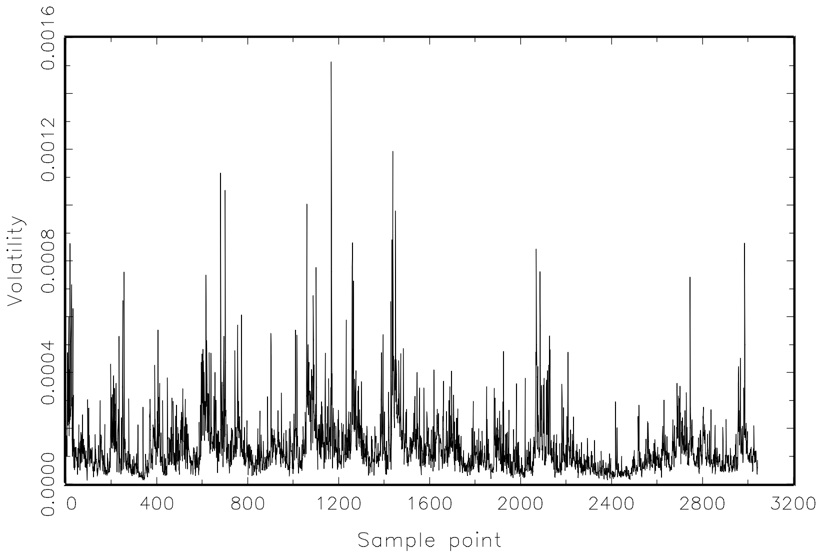


Figure 2b: DM/USD returns
Conditional 20th, 50th, 80th percentiles of volatility
ARCH representation

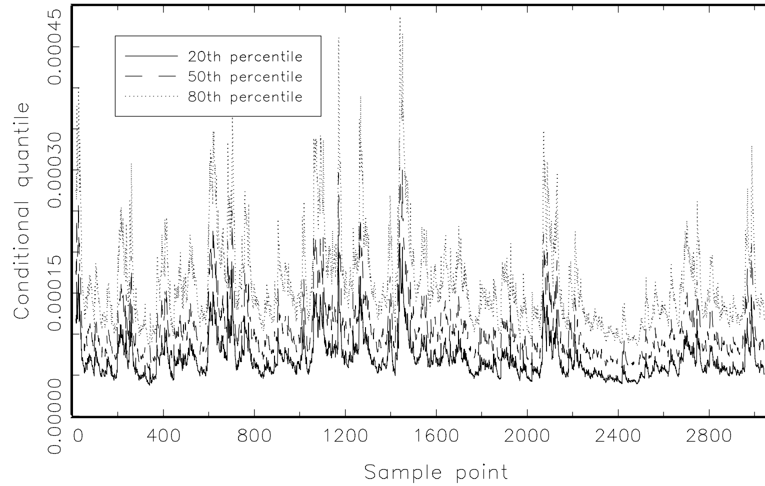


Figure 2c
Ratio of conditional 80th percentile to conditional 20th percentile
ARCH representation

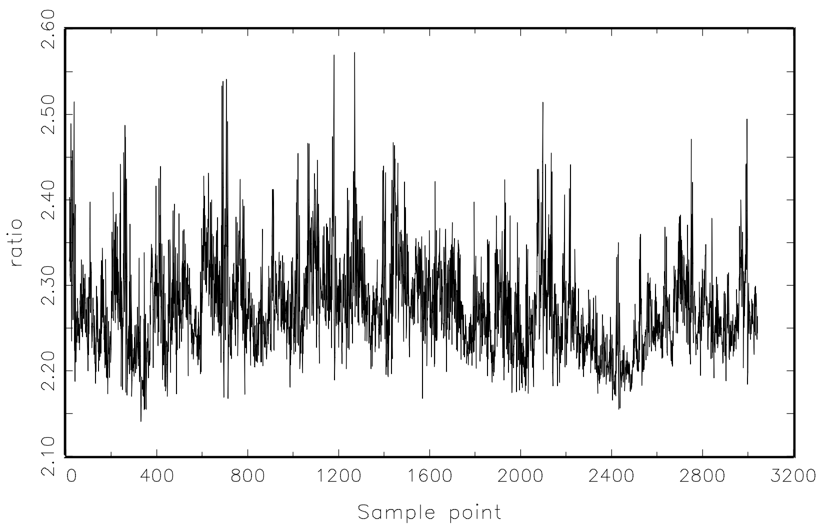


Figure 2d
Ratio of conditional 90th percentile to conditional 10th percer
ARCH representation

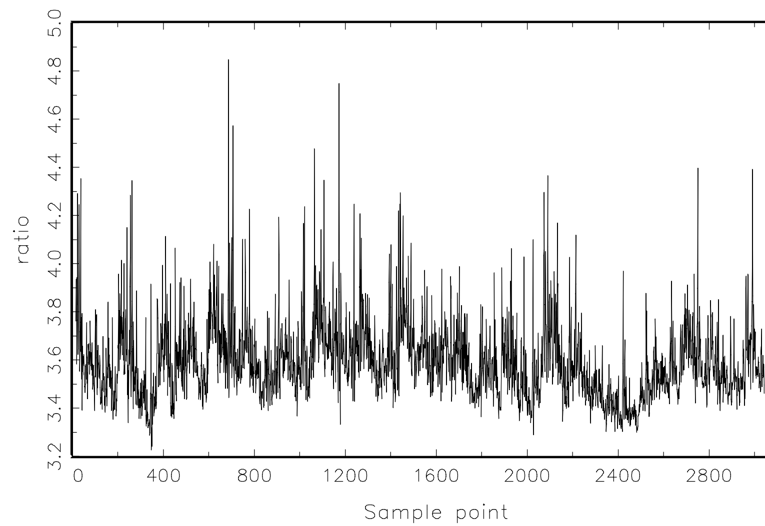


Figure 3a: Yen/ USD returns
Realized volatility, five-minute returns

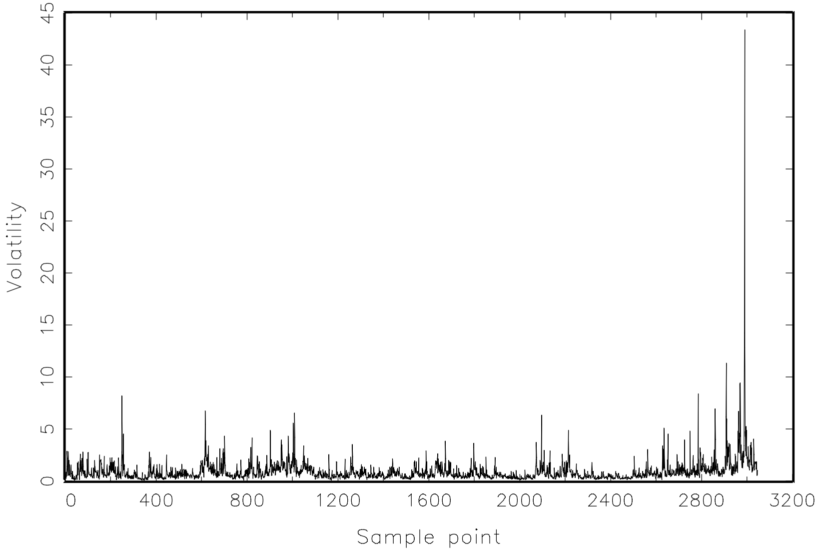


Figure 3b: Yen/ USD returns
Conditional 20th, 50th, 80th percentiles of volatility
ARCH representation

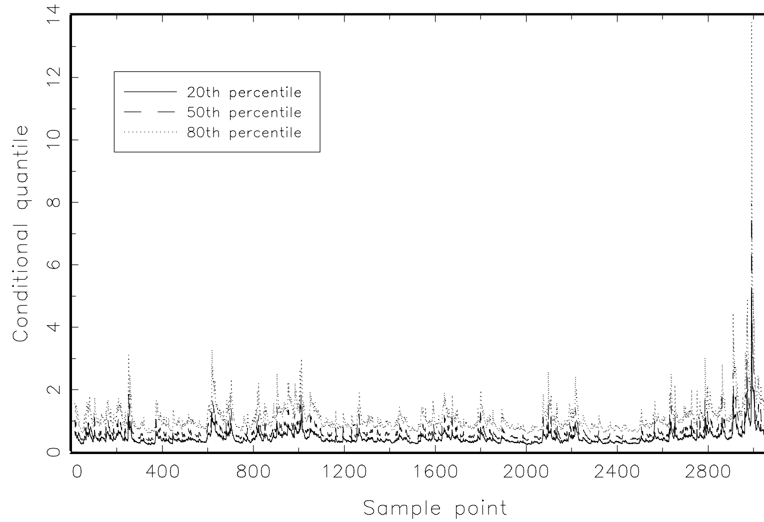


Figure 3c
Ratio of conditional 80th percentile to conditional 20th percentile
ARCH representation

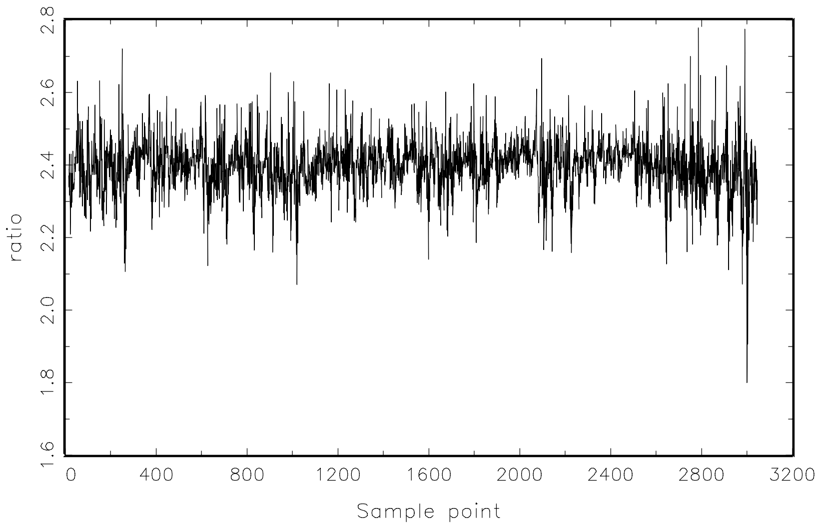
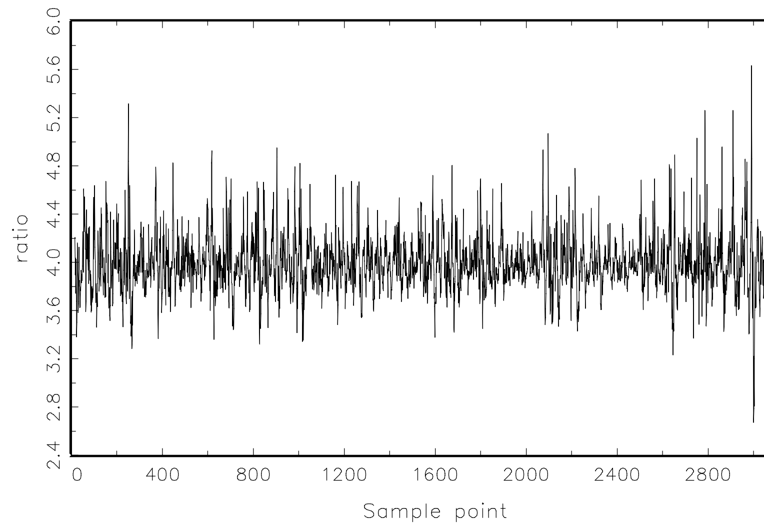


Figure 3d
Ratio of conditional 90th percentile to conditional 10th percer
ARCH representation



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