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Knowledge acquisition within an organization: How to retain a knowledge worker using wage profile and non-monotonic knowledge accumulation

Ngo Van Long [*], *Antoine Soubeyran* [†], *Raphael Soubeyran* [‡]

Résumé/abstract

Dans ce papier, nous considérons un problème d'accumulation de connaissances dans une organisation. Nous partons de la théorie du capital humain lancée par Becker (1962, 1964) et considérons une organisation qui ne peut pas empêcher un employé de quitter et d'utiliser la connaissance à l'extérieur de l'organisation. Nous montrons comment l'employeur manipule de façon optimale le sentier d'accumulation de connaissances et choisit un profil de salaire pour atténuer le problème d'engagement. Nous montrons que l'accumulation de connaissances est retardée : la fraction de temps alloué à la création de connaissances est la plus haute au premier stade de la carrière, puis elle tombe progressivement, ensuite, elle monte de nouveau, avant de tomber finalement vers le zéro. Nous déterminons l'effet de la spécificité de connaissances. Nous discutons aussi la forme des profils de salaire optimaux, le rôle du niveau de connaissance initial et du rôle du fait du taux d'actualisation.

Mots clés : Capital humain, hold-up, contrat.

In this paper, we consider a knowledge accumulation problem within an organization. We depart from the human capital theory initiated by Becker (1962, 1964) and consider an organization that cannot prevent the worker from quitting and using the knowledge outside the organization. We study how the employer optimally distorts the knowledge accumulation path and chooses a wage profile in order to mitigate the commitment problem. We show that knowledge accumulation is delayed: the fraction of working time allocated to knowledge creation is highest at the early career stage, falls gradually, then rises again, before falling finally toward zero. We determine the effect of a change in the severity of the enforcement problem (or the specificity of knowledge). We also discuss the form of the optimal life-cycle wage profiles, the role of the initial knowledge level and the role of discounting

Key words: *Human capital,hold-up, contract*

Codes JEL *:* J2

[*] CIRANO, and Department of Economics, McGill University. Email: ngo.long@mcgill.ca
[†] Aix-Marseille University (Aix-Marseille School of Economics), CNRS & EHESS. Email: Antoine.soubeyran@univ_amu.fr
[‡] INRA, UMR1135 LAMETA, F-34000 Montpellier, France. Email: soubeyra@supagro.inra.fr

Introduction

One of the main issues in leading knowledge workers is how to retain them in the firm, as monetary incentives alone are often not sufficient. The nature of the tasks given to knowledge workers are an important incentive to retain them in the firm (Prince, 2011). Companies often propose new tasks to their knowledge workers, and academic departments of research-oriented universities assign teaching loads and administrative responsibilities (that constraint research time) that vary along the career path.[1]

The human capital theory initiated by Becker (1962, 1964) and Mincer (1958, 1962, 1974) has offered a rich analysis of an individual's life cycle investment in human capital. One of the main results of this literature states that human capital investments are undertaken at the early stage of the career because workers have then a longer period of time over which they can benefit from the return of their investments. Becker's focus on the investment demand side has been complemented by supply-side considerations offered by Ben-Porath (1967), who assumed that an individual must allocate a fraction of his human capital as an input, to be combined with purchased inputs, in his investment in human capital. Ben-Porath showed that this fraction is changing over time, and generally it becomes smaller and smaller as the individual approaches the retirement age.[2] Both the Becker mechanism and the Ben-Porath mechanism predict that for any individual the investment in human capital declines over time.

In this paper, we consider a different mechanism that, unlike the other two previous mechanisms, is capable of producing a non-monotone path of investment in human capital. The driver behind our result is the role of the employer, who actively offers the knowledge workers time-varying financial and non-financial incentives to stay with the firm. In the first best situation, where the employer and the knowledge worker can sign binding contracts, we show a result similar to the standard human capital theory, i.e., the share of time allocated to knowledge creation decreases over time. In the second-best situation, where the knowledge worker can leave the organization and use the accumulated knowledge to earn income outside the organization, we show that the time path of investment in human capital can be non-monotone. At the beginning of her career, the worker is asked to spend a large share of her time to knowledge creation, but gradually she is asked to allocate an increasing share of time to routine tasks. Around the middle of her career, this trend reverses and the employer allows the knowledge worker to devote more and more time to knowledge creation. Toward the end of her career, the trend reverses again and the knowledge worker is asked to perform more and more routine tasks.

Our main result fits well with the existing evidence on life-cycle human capital accumulation profiles. Human capital accumulation being difficult to measure, the empirical literature provides mainly documentation of formal on-the-job training. The available evidence is consistent with our result that human capital

[1] For instance, Google's engineers can work 20% of their time on independent projects.

[2] Ben-Porath made the distinction between observed earnings and earnings net of all investment costs: the former are always higher, change slowly, and peaks at an earlier date than the latter.

accumulation is not concentrated at the beginning of the career. Indeed, Loewenstein and Spletzer (1997) show that delayed formal training (i.e. an increasing quantity of formal training over time) is frequent. This empirical result suggests that human capital investments are not undertaken without delay, at the early stage of the career. In contrast to the presumption of concentrated accumulation of human capital at the beginning of the career, our main result shows that firms prefer some delay in human capital investments in order to retain the worker (this point will be further discussed in section 5).

After establishing our main result, we discuss the effect of a change in the severity of the enforcement problem (or the specificity of knowledge). We show that an increase in the enforcement problem (or a decrease in the degree of firm-specificity of knowledge) decreases the share of working time allocated to knowledge creation and decreases cumulated knowledge. Somewhat surprisingly, an increase in the difficulty of enforcement can increase or decrease the life-time income of the knowledge worker.

Consistent with Ben-Porath (1967), our model is able to generate the main feature of age-earnings profiles (e.g. Heckman 1976, Rosen 1976, and Murphy and Welch 1990), namely, the experience-earnings profile is a hump-shaped function of experience.

We also discuss the returns to schooling in our model (see Heckman et al. 2003 and Belzil 2007 for surveys of the literature). We show that our main result is robust to an increase in the initial knowledge level. And, consistently with recent evidence that differences in lifetime earnings are mainly due to initial conditions (see Hugget et al. 2011), we show that an increase in the initial knowledge level has a positive effect on earnings.

The paper is organized as follows: Section 1 presents the model and notations; in Section 2 we derive the first best knowledge accumulation path; in Section 3, we show how an employer retains a knowledge worker using wage profile and delayed knowledge accumulation. We also discuss the shape of the optimal wage profile and show how a change in the severity of the enforcement problem affects our results. In Section 4, we focus on the role of the initial knowledge level, the role of the discount rate, and show that our results remain valid in the case of a strictly concave utility function. Section 5 discusses the link between our model and the empirical evidence. In Section 6, we relate our model to several strands of the literature. Section 7 concludes.

1 Model and notations

The time horizon T represents the maximal career duration of the employee. Each period, the employee is endowed with a fixed working time (normalized to 1) to be split between time spent to create knowledge, $k(\tau)$, and time spent on the usual task, $l(\tau)$. The time constraint of the employee is $k(\tau) + l(\tau) = 1$. The employee's cumulated knowledge at any time t is $K(t) = \int_0^t k(\tau)\, d\tau$.

Inside the firm, the profit at time τ generated by the employee's cumulated knowledge is denoted by

$\pi(K(\tau))$ with $\pi(0) = 0$, $\pi' > 0$ and $\pi'' \leq 0$.

We make a natural assumption concerning outside options: if the employee leaves the firm at time t_l, her expected future revenue (per period) is an increasing function of the stock of knowledge cumulated up to t_l. Moreover, following Becker (1962), we distinguish two categories of human capital, specific and general human capital. Specific human capital is related to a specific firm's products or services whereas general human capital can be used in a large range of different firms. Accordingly, the employee's earning outside the firm will be $\hat{\pi}(K(t_l))$, where we assume that $\hat{\pi}(K) \equiv \pi(\beta K)$, with $\beta \in (0,1)$.[3] In other words, knowledge is more useful inside the firm than outside and $1-\beta$ represents the degree of specificity of knowledge. The employee who quits at t_l receives $\pi(\beta K(t_l))$ each period of time $\tau > t_l$ outside the firm, using the knowledge cumulated up to t_l. After the employee quits, the firm obtains a constant value (normalized to 0) each remaining period of time.

As long as the employee remains with the firm, it earns the profit $\pi(K(\tau))$, and an additional amount $v(l(\tau))$ if the employee spends a fraction $l(\tau)$ of her time endowment on routine tasks. Assume that for all $l \in [0,1]$, $v(l) \geq 0$, non-decreasing, and strictly concave. The employee and the employer have the same discount rate $r \geq 0$. Under our assumptions, the surplus of the relationship at time τ is $\pi(K(\tau)) + v(l(\tau)) - \pi(\beta K(\tau)) \geq 0$.

At time $t = 0$, the employer and the employee sign a contract. The contract specifies the fraction $k(\tau)$ of working time that must be devoted to knowledge creation, the remaining fraction $l(\tau)$ being allocated to the routine task, $\tau \in [0,T]$, and a wage profile represented by $w(\tau)$ for $\tau \in [0,T]$. Thus, if the employee quits at time t_l, the cumulated payoff of the employee over the lifetime horizon is:

$$V_w = \int_0^{t_l} e^{-r\tau} w(\tau)\, d\tau + \int_{t_l}^{T} e^{-r\tau} \pi(\beta K(t_l)) d\tau,$$

and the cumulated payoff of the employer is[4]

$$V_f = \int_0^{t_l} e^{-r\tau} \left[\pi(K(\tau)) + v(l(\tau))\right] d\tau - \int_0^{t_l} e^{-r\tau} w(\tau)\, d\tau.$$

Notice that our model does not include other possible payoffs such as the knowledge worker's ego-rent, or her pleasure obtained from knowledge creation. Adding such factors would make the model more cumbersome, without changing our main results.

[3] Our main results are not affected if we instead consider the more general assumption that $\hat{\pi}$ is an increasing and concave function of K, such that $\hat{\pi}(K) \leq \pi(K)$.

[4] Despite we assume additivity, notice that a multiplicative form of the instantaneous payoff is compatible with our model. If one chooses $\pi(.) \equiv v(.) \equiv \ln(.)$, the instantaneous payoff from working $\pi(K) + v(l)$ can be written in the multiplicative form $\ln(K \times l)$.

2 First best solution

If the employee leaves the firm at time $t_l \leq T$, the joint surplus over the time horizon T is given by

$$J(t_l) = \int_0^{t_l} e^{-r\tau} \left[\pi \left(K(\tau)\right) + v\left(l(\tau)\right)\right] d\tau + \int_{t_l}^{T} e^{-r\tau} \pi(\beta K(t_l)) d\tau,$$

where $k(\tau) + l(\tau) = 1$. The opportunity cost of time devoted to the knowledge task can be written as $v(l(\tau)) = v(1 - k(\tau))$. The joint surplus optimization programme can then be written as

$$\max_{0 \leq t_l \leq T,\ 0 \leq k(\tau) \leq 1} \left\{ J(t_l) = \int_0^{t_l} e^{-r\tau} \left[\pi \left(K(\tau)\right) + v\left(1 - k(\tau)\right)\right] d\tau + \int_{t_l}^{T} e^{-r\tau} \pi(\beta K(t_l)) d\tau \right\},$$

where $0 \leq k(\tau) \leq 1$,

and subject to

$$\dot{K}(\tau) = k(\tau),$$

and,

$$K(0) = 0 \text{ (given)}.$$

Clearly, since $\pi(\beta K) \leq \pi(K)$ for all $K \geq 0$ and since $v(1-k) \geq 0$ for all feasible $k \in [0, 1]$, the solution of the joint surplus maximization problem displays the plausible property that the two parties stay together until T. Another property of the solution concerns the fraction of time devoted to knowledge creation: it is decreasing over time. This is described in the following proposition.[5]

Proposition 1 [First best allocation of time between tasks]: *The optimal duration of the relationship is $t_l^* = T$ and the optimal (interior) splitting of working time is such that the share of time allocated to knowledge accumulation is decreasing whereas the share of time allocated to the routine task is increasing ($\dot{k} < 0$ and $\dot{l} > 0$).*

The joint surplus maximization requires that the share of working time allocated to the knowledge task decreases over time and the share of time allocated to the routine task increases over time. In other words, the share of the routine task is gradually becoming dominant. Knowledge is accumulated mainly at the beginning of the career because the sooner the investment in the knowledge task, the larger the cumulative benefits. When the employee approaches the end of the horizon, knowledge accumulation becomes less attractive because there is less remaining time to exploit knowledge. Indeed, the shadow price of knowledge, ψ, decreases through time, $\psi(t) = \int_t^T \pi'(K(\tau)) e^{-r\tau} d\tau$ and $\dot{\psi}(t) = -\pi'(K(t)) e^{-rt} < 0$.

Let us illustrate this result with an example:

[5] All proofs are relegated to the Appendix.

Example 1: Let us specify the functions as follows. $\pi(K) = AK$ with $A > 0$ and $v(l) = l - \frac{1}{2}l^2$. We also assume that $AT \leq 1$ which ensures the existence of an interior solution. (For the Figures, we set $T = 1$, $A = 1/2$). In this example, we have $k^*(\tau) = A(T - \tau)$ and $l^*(\tau) = 1 - A(T - \tau)$.

[INSERT FIGURES 1a,b]

The first best solution is based on the assumption that both parties can commit to continuing their relationship until T. However, since the employee is accumulating knowledge, the value of her outside income ($\pi(\beta K)$) is increasing over time, hence if the employee is free to quit at any time, she would have an incentive to quit unless the employer promises her sufficient reward for staying with the firm. This reward can come in two forms: a monetary reward, such as a time profile of salary that evolves with seniority, or a prospect of accelerated increase in human capital in the future (which tilts the time path of her outside option toward the future). The next section investigates this issue.

3 How to retain a knowledge worker using wage profile and non-monotonic knowledge accumulation

We now consider the situation where the knowledge worker can quit at any time $t \in (0, T)$, taking with her the knowledge that she has acquired while being employed. We derive the properties of the optimal splitting of working time when the employer designs a scheme that prevents the knowledge worker from quitting the firm before the retirement age T. The firm has two instruments to retain the worker: it prescribes a time path of allocation of working time between knowledge creation and the routine task, and it gives a monetary payment to the knowledge worker. An interesting feature of our model is the optimal trade off (from the firm's vantage point) between these two instruments. For simplicity, we assume that both the employer and the employee do not discount the future, $r = 0$. We will relax this assumption in sub-section 4.2 and show that our main result is not affected: the optimal splitting working time remains non-monotonic.

The knowledge worker receives a non-negative wage, $w(\tau)$, at time τ. We assume that the firm cannot ask the worker to post a bond and it cannot ask her to pay any compensation once she has left. From time t, if the worker stays with the firm up to T she enjoys a cumulated wage $W(t) \equiv \int_t^T w(\tau)\, d\tau$. If she chooses to leave the firm at time t, her payoff for the remaining working life is $R(t) \equiv \int_t^T \pi(\beta K(t)) d\tau = (T - t)\,\pi(\beta K(t))$. It is important to notice that the value of her outside option, denoted by $R(t)$, evolves through time. The worker will not quit before the time horizon T if the firm offers her a contract such that the following non-quitting constraint is satisfied:

$$W(t) \geq R(t) \text{ for all } t \in [0, T]. \tag{1}$$

The problem to be solved by the employer can then be written as follows:

$$\max_{0\leq k(\tau)\leq 1,\ 0\leq w(\tau)} \int_0^T \left[\pi(K(\tau)) + v(1-k(\tau)) - w\left(\tau\right)\right] d\tau \tag{2}$$

subject to

$$\dot{K}(\tau) = k(\tau),$$

$$K(0) = 0 \text{ (fixed)},$$

and the non-quitting constraint (1).

3.1 Non-monotonic working time allocation

The firm could maintain the first best time allocation scheme and keep the employee by promising each time t a sufficiently large future monetary transfer, $W(t)$, on the condition that the employee stays with it until T. But such a policy would be too expensive. The total wage payment, $W_0 = \int_0^T w(\tau)\, d\tau$, can be much reduced if the employee has less valuable outside option, and this can be achieved if the firm reduces her path of accumulated human capital below the first best path. While this is intuitively plausible, what is not clear is whether it would be optimal for the firm to design a non-monotone path of working time allocation. Let us turn to this issue. Does the possibility of introducing a non-monotone path of working time allocation help the firm to keep the worker, while trimming down the total wage payment? This sub-section provides an answer to this question.

Let us write the necessary conditions for this optimal control problem. Let $\psi(t)$ and $\rho(t)$ be the co-state variables associated with the state variables $K(t)$ and $W(t)$ respectively. Let $\lambda(t)$, $\mu(t)$, $\alpha(t)$ and $\varphi(t)$ be the multipliers associated with the inequality constraints $k(t) \geq 0$, $1-k(t) \geq 0, w(t) \geq 0$ and $W(t) \geq R(t) = (T-t)\,\pi(\beta K(t))$. Since the "initial" W_0 is not exogenously given, but is an object of choice, it is convenient to re-write the objective function (2) as

$$\max_{W_o,\ 0\leq k(\tau)\leq 1,\ 0\leq w(\tau)} \int_0^T \left[\pi(K(\tau)) + v(1-k(\tau))\right] d\tau - W_0 \tag{3}$$

Note that

$$W(t) \equiv W_0 - \int_0^t w(\tau) d\tau \Longrightarrow \dot{W}(t) = -w(t).$$

Then, we can define the Hamiltonian H and the Lagrangian $\pounds$ as follows

$$H = \pi(K) + v(1-k) + \psi k - \rho w,$$

$$\pounds = H + \lambda k + \mu\left[1-k\right] + \alpha w + \varphi\left[W - R\right].$$

Looking at a solution such that k is interior, the necessary conditions include

$$\frac{\partial \pounds}{\partial k} = -v'(1-k) + \psi = 0, \tag{4}$$

$$\frac{\partial \pounds}{\partial w} = -\rho + \alpha = 0, \tag{5}$$

$$\alpha \geq 0,\ w \geq 0,\ \alpha w = 0, \tag{6}$$

$$\varphi \geq 0,\ W - R \geq 0,\ \varphi (W - R) = 0, \tag{7}$$

$$-\dot{\psi} = \frac{\partial \pounds}{\partial K} = \pi' (K) - \varphi \beta (T - t) \pi' (\beta K), \tag{8}$$

$$-\dot{\rho} = \frac{\partial \pounds}{\partial W} = \varphi. \tag{9}$$

The following transversality conditions are also necessary:

Since K_T and W_0 are free, but W_0 appears linearly as a cost in the objective function (3), the following conditions are also necessary

$$\psi(T) = 0, \tag{10}$$

$$\rho(0) = 1. \tag{11}$$

Since there is no restriction on $W(T)$, we have the transversality condition

$$\rho(T) = 0. \tag{12}$$

The following interpretations of the shadow price $\rho(t)$ may be useful. The firm offers the employee the promise that the present-value of all wage payments after date t is $W(t)$, which will be paid out gradually, but which the employee must forfeit if she quits. Since the firm must honor its contract, the monetary cost of this promise is W_0. It is as if the firm must, at date 0, put in a "trust account" the total life-time wage W_0. At any time $t > 0$, the amount of funds that remains in this trust account is $W(t) \leq W_0$. From the firm's point of view, the remaining balance $W(t)$ is a valuable stock, because it prevents the employee from quitting. That is why $\rho(t) \geq 0$, i.e., the shadow price of this stock is non-negative. Equation (11) states that, at time 0, the optimal choice W_0 is such that the shadow price of this stock, $\rho(0)$, is equal to the marginal cost of "purchasing" this stock, which is 1 (the derivative of the term W_0 in the objective function (3) with respect to W_0).

Proposition 2 [Second best allocation of time between tasks]: *The second best program is very different from the first best program, and can be characterized as follows. If* $k(t) > 0$ *over* $[0, T)$,

(i) There exists ***at least one time interval*** (t', t'') *over which* $k(\tau)$ *is* ***increasing***, i.e. *the employer promises a phase of acceleration of human capital accumulation in order to induce the agent to stay longer.*

(ii) There exists $t_b < T$, *such that over the interval* $[t_b, T]$, $k(t)$ *will be falling* ($\dot{k} < 0$).

(iii) There exists $t_a > 0$, *such that over the interval* $[0, t_a]$, $k(t)$ *will be falling* ($\dot{k} < 0$).

(iv) The necessary conditions are sufficient.

The second best allocation scheme offered by the firm differs from the first best because, knowing that the worker has the freedom to quit at any time, it has to propose a contract in which the path of allocation of working time between tasks and the wage profile are suitably designed to counter the worker's incentives to leave the firm. The firm distorts the path of knowledge accumulation in order to reduce the total wage payment it has to make to retain the worker. The optimal path of knowledge accumulation exhibits an intermediate phase where the proportion of time allocated to knowledge accumulation increases.

The main difference with the first best case lies in $\beta > 0$. Indeed, if $\beta = 0$, the agent's outside option is 0 and then the employer can use the first best path of knowledge accumulation. $\beta > 0$ affects the shadow price of knowledge through the non quitting constraint. As time passes, the shadow price of knowledge changes according to the law of motion $\dot{\psi}(t) = -\pi'(K(t)) + \varphi\beta(T-t)\pi'(\beta K(t))$. The first term, $-\pi'(K(t))$, is negative, i.e. a tendency for $\psi(t)$ to fall, because the profit at time t has been earned (i.e., the knowledge stock becomes less valuable because there is now a shorter remaining time horizon to exploit it). The second term, $\varphi\beta(T-t)\pi'(\beta K) \geq 0$, is non negative. It represents the opposite tendency for the shadow price to rise, as the future knowledge provides the temptation for the employee to stay longer, to improve her outside option. When $\beta \to 0$, the dynamics of the shadow price is the first best one. The possibility that the employee leaves the firm (when $\beta > 0$) increases the future shadow price of knowledge and then provides incentives to the firm to delay knowledge accumulation.

The intuition of the proof is the following. If the non-quitting constraint was never binding over the horizon, the employer was paying too much. So he would reduce wages. Hence the optimal W_0 must be such that the non-quitting constraint is binding over at least one interval, i.e., $W(t) = R(t)$ for all t in some interval $[t_a, t_b]$. The shadow price $\rho(t)$ of the remaining earnings $W(t)$, is decreasing, $\dot{\rho}(t) \leq 0$. Hence, when the constraint is binding, the employer has an incentive to postpone wages and thus the remaining earnings and the outside option are constant over this interval, $\dot{W}(t) = -w(t) = 0 = \dot{R}(t)$. The only way to keep the outside option constant is to choose an increasing rate of knowledge accumulation, $\dot{k} > 0$ over this interval.

Remark: *From equations (9), (11) and (12), we have*

$$\int_0^T \varphi(t)dt = \rho(0) - \rho(T) = 1 \tag{13}$$

which implies that $\varphi(t) > 0$ over at least some time interval $[t_a, t_b]$. In other words, the non-quitting must be binding over some time interval.

Let us illustrate the result with an example:

Example 2 [Second best allocation of time between tasks]: Let us specify the functions as follows:[6] $\pi(K) = AK$ with $A > 0$ and $v(l) = l - \frac{1}{2}l^2$ (note that $v'(1) = 0 < A$, hence it is never optimal to set $l = 1$). Assuming that there is just one interval $[t_a, t_b]$ over which the non-quitting constraint is binding (we know from point (iv) of Proposition 2 that this solution is optimal), the optimal path of knowledge accumulation and the optimal fraction of working time devoted to the knowledge task can be illustrated with Figures 2.a,b,c ($T = 1$, $A = 1/2$, $\beta = 0.8$).

[6] The resolution of the firm's problem under this specification is relegated to Appendix B.

[INSERT FIGURES 2a,b]

The intuition behind Proposition 2 can be illustrated in this example. Figure 2.a shows the first best (dotted curve) and the second best path of the share of time spent to create knowledge (Figure 2.b shows the cumulated knowledge curves). The second best time spent to create knowledge is lower than the first best, because the employer does not receive 100% of the return of his "investment" in knowledge accumulation. Indeed, because of the hold-up problem, the employer has to share the returns of his investment in knowledge with the employee (the classical under-investment argument in hold-up models). The new feature is that the employer will permit an increase in the time spent to create knowledge during the employee's mid-career. In the final phase of her career, the example shows that the first best and the second best allocations of time coincide.

[INSERT FIGURE 2c]

In this example, the outside option of the knowledge worker is given by:

$$R(t) = (T - t) A\beta K(t).$$

Now let us compare how the outside option evolves through time when the employer uses the first best path of working time allocation and when he uses the second best path (see Figure 2.c): The dotted curve (respectively, the continuous curve) represents the outside option of the knowledge worker when the employer uses the first best (respectively, second best) allocation of working time. In the second best situation, the firm has to offer a cumulated wage that fulfils condition (1). If it uses the first best path, it has to pay a total amount $W(0) = \int_0^T w(\tau)\, d\tau \simeq 0.08$ to the worker. According to Proposition 2, this is not optimal. If instead the firm uses the second best path, the outside option curve becomes flatter and it only has to pay less than 0.02 to the knowledge worker.

3.2 Optimal wage profiles

The empirical literature has shown that the Ben-Porath (1967) model matches age-earning profiles (e.g. Heckman 1976 and Rosen 1976). The main feature of the empirical experience-earning profile is that the earning function is a hump-shaped function of experience.

The following result shows that our model is compatible with this empirical evidence (we still consider that $r = 0$).

Proposition 3 [Hump-shaped life cycle wage profile]: *There always exists an optimal hump-shaped experience-earnings profile.*

The form of the optimal wage profile is constrained by the non-quitting constraint, i.e. the remaining earnings $W(t)$ have to be greater than the outside option $R(t)$ at any time t. In terms of the necessary conditions (8) and (9), the dynamics of the shadow price of knowledge ψ and the shadow price of the remaining wages ρ are linked through the opportunity cost of retaining the employee, which is represented by φ. The shadow price of the remaining earnings is non increasing, $\dot{\rho} = -\varphi \leq 0$, because one unit of wage w paid at time t' rather than t with $t \leq t'$ does not affect $W(t)$ but increases $W(t')$.

We know that the shadow price of the remaining wages is strictly positive, $\rho > 0$, over $[0, t_b)$. Hence, the employer has an incentive to postpone the earnings of the employee and then $w(t) = 0$ over $[0, t_b)$. In the last phase $[t_b, T]$, the employer can spread the total earnings, $W_0 = W(t_b) = R(t_b)$, such that the non quitting constraint is fulfilled. There are then many optimal wage profiles. The first polar wage profile is such that the employer offers all the earnings $W(t_b)$ at the end of the horizon and the second polar wage profile is such that the employer binds the non-quitting constraint at each time, i.e. $W(t) = R(t)$ for all $t \in [t_b, T]$.[7] Then, the optimal wage profiles are then such that the remaining earnings function is:

$$W^*(t) = \begin{cases} R^*(t_b) \text{ for } t \in [0, t_b] \\ s(t) R^*(t_b) + (1 - s(t)) R^*(t) \text{ for } t \in [t_b, T] \end{cases}$$

where $s(t) \in [0, 1]$.

Using $w(t) = -\dot{W}(t)$, we can deduce the optimal wage:

$$w^*(t) = \begin{cases} 0 \text{ for } t \in [0, t_b] \\ -\dot{s}(t)[R^*(t_b) - R^*(t)] - (1 - s(t)) \dot{R}^*(t) \text{ for } t \in [t_b, T] \end{cases}$$

Clearly, the function $s(t)$ is not uniquely determined. In the proof of Proposition 3 we show that we can choose a sigmoid function $s(t)$ such that the optimal wage profile is hump-shaped and Example 3 provides a numerical illustration. Figures 3.a and 3.b illustrate the result of Proposition 3. The non quitting constraint states that the remaining earnings at any time t, $W(t) \equiv \int_t^T w(\tau) d\tau$, have to be larger than the optimal outside option $R^*(t) = (T - t)\pi(\beta K^*(t))$. It is possible to find W such that it is larger than the optimal outside option curve $R^*(t)$.

Example 3: As in Example 2, we specify the functions as follows:[8] $\pi(K) = AK$ with $A > 0$ and $v(l) = l - \frac{1}{2}l^2$. We also focus on the optimal solution such that k increases over only one interval $[t_a, t_b]$. We set $T = 1$, $K(0) = 0$, $A = 1/2$, and $\beta = 0.8$.[9]

[INSERT FIGURES 3a,b]

Figures 3a and 3b display hump-shaped wage profiles that resemble the bell curve of a normal distribution. This is because we use a sigmoid function $s(t)$ and assume a low level of initial human capital. As example 3' (below) shows, it is possible to generate wage profiles that look like those in the data, in particular, one

[7] Then $w(t) = -\dot{R}^*(t) = -\pi(\beta K^*(t)) + (T - t)\pi'(\beta K^*(t))\beta k^*(t)$.

[8] The resolution of the firm's problem under this specification is relegated to Appendix B.

[9] And the additional parameters are set to: $\sigma = 1.2$, $\lambda = 8$. See Appendix A for the parametric function used in this example (proof of Proposition 3).

that is concave everywhere, with the biggest increases in wages occuring at the beginning of a career, such as Figure 2 in Murphy and Welch (1990).

Example 3': We set $s(t) = 0.2t + 0.8, r = 0, K(0) = 1, A = 1$,and $\beta = 0.8$.

[INSERT FIGURE 3c]

3.3 Enforceability and specificity of knowledge

The most important parameter of our model is β and it can be interpreted in two different ways. First, β determines the degree of severity of the enforcement problem. Indeed, if $\beta = 0$, the employee has no incentives to quit the firm and the employer can then implement the first best path of knowledge accumulation without offering any positive wage to the employee. If $\beta > 0$, the employer has to take into account the non-quitting constraint and to distort the knowledge accumulation path. Second, $1 - \beta$ measures the degree of specificity of knowledge ($\beta = 0$ being full specificity and $\beta = 1$ being full general knowledge). The latter interpretation is in line with Becker (1962) who distinguishes two categories of human capital, specific and general human capital. Specific human capital is related to a specific firm's products or services, whereas general human capital can be used in a large range of different firms.

More specifically, we study the effect of a change in β on the outcomes of the model. Since there exist many optimal wage profiles, we analyse the effect of β on the optimal total earnings, W_0, rather than on the whole wage profile, $w(t)$. We then study the effect of β on the outcomes $(k(t), K(t), W_0)$ and also on the length of the phase $[t_a, t_b]$. We concentrate on the following specification of the general model: we assume, as in examples 1, 2 and 3 that the profit generated by knowledge inside the firm is proportional to the accumulated amount of knowledge, $\pi(K) = AK$. The value of the routine task is specified as $v(l) = l - \frac{1}{2}l^2$. Using this specification and focusing on the solution with only one phase where k increases (again see Appendix C for sufficiency), we can show the following result:

Proposition 4 [Enforceability and specificity of knowledge]: *As the severity of the enforcement problem decreases and/or knowledge becomes more specific (β decreases):*

(i) The phase of increasing fraction of time devoted to the knowledge task becomes shorter. Formally, $|t_b - t_a|$ *decreases when* β *decreases.*

(ii) Both the fraction of time devoted to the knowledge task and the total amount of knowledge accumulated at any time increase. Formally, both $k(t)$ and $K(t)$ *increase when* β *decreases, for all* $t \in [0, T]$.

(iii) The change in the total wage of the knowledge worker, W_0*, is ambiguous.*

The first two results (i) and (ii) are quite intuitive. When knowledge becomes more firm-specific, the value of the outside option of the knowledge worker becomes smaller and it becomes easier to retain her, i.e. the situation is closer to the first best situation. Then the duration of the phase where k increases becomes smaller. (Note that there is no such phase in the first best case, which is incentive compatible in the situation where $\beta = 0$). The total amount of knowledge accumulated over the whole horizon also increases. When β decreases, the severity of the hold-up problem is reduced, the "investment" k (and so K) then increases and becomes closer to the first best path (see Figures 4.a and 4.b for a numerical illustration).

The result reported in point (iii) can be explained as follows. An increase in the specificity of knowledge has two effects on the outside option, which work in opposite directions. On the one hand, the more specific the knowledge, the lower the value of the outside option of the knowledge worker, and this allows the employer to offer her a lower total wage. On the other hand, when knowledge becomes more specific, the employer can afford to leave her more time to accumulate knowledge. Hence, the amount of accumulated knowledge is larger, which in turn increases the knowledge worker's outside option. These two opposite effects explain the ambiguous impact of knowledge specificity on the worker's wage.

The following example illustrates these results.

Example 4: As in example 2, we set $T = 1$ and $A = 1/2$.

Using the parametric hump-shaped wage profile used to plot Figure 3.a, Figure 4.c illustrates this optimal wage profile for various values of β. We can see that when β changes, the optimal wage profile is affected. The larger β is, the greater the cumulated wage (the greater the area below the wage profile curve). Notice that the wage profile for $\beta = 0$ is $w(t) = 0$ for all $t \in [0, T]$, because the worker has no outside option ($R(t) \equiv 0$).

[FIGURES 4a,b,c]

4 Extensions: initial knowledge level, discounting, and concave utility

4.1 Role of initial knowledge level

The seminal contribution of Mincer (1974) has initiated many empirical works regarding the returns to schooling (see Heckman et al. 2003 and Belzil 2007 for surveys of the literature). More recently, Hugget et al. (2011) have shown that differences in lifetime earnings are mainly due to initial conditions. In this section, we show how a change in the initial knowledge level ($K_0 \equiv K(0)$) affects our main results.

We concentrate on our main example. The profit function is linear, $\pi(K) = AK$. The degree of knowledge specificity is denoted $1 - \beta \in (0, 1)$. The value of the routine task is specified as $v(l) = l - \frac{1}{2}l^2$. Using this specification and focusing on the solution with only one phase where k increases (again see Appendix C for sufficiency), we can show the following result:

Proposition 5 [Initial knowledge level]: *As the initial stock of knowledge K_0 increases,*

(i) The fraction of time devoted to the knowledge task is shifted downwards in the first phase, then upwards in the second phase, and is not affected in the third phase. Formally, when K_0 increases, $k(t)$ becomes greater for each t in $[0, t_a]$, smaller for each t in $[t_a, t_b]$, and is not affected over $[t_b, T]$.

(ii) The cumulated knowledge always increases. Formally, $K(t)$ becomes greater at each t when K_0 increases.

(iii) The phase of increasing fraction of time devoted to the knowledge task becomes longer. Formally, $|t_b - t_a|$ increases when K_0 increases. However, both t_a and t_b decrease when K_0 increases.

(iv) The total earnings are greater and the phase where the wage profile is flat is shorter. Formally, W_0 increases when K_0 increases. Moreover, $w(t) = 0$ over $[0, t_b]$ and t_b decreases when K_0 increases.

When the initial knowledge of the employee increases, her outside option, $R(t) = (T - t)\,\pi(\beta K(t))$, is greater and reaches its falling stage at an earlier time. A larger K_0 leads to a larger $\pi(\beta K(t))$, and then t_b is smaller (see Figure 5.a). The employer has an incentive to reduce the share of time devoted to knowledge creation at the beginning in order to limit the increase of the outside option of the employee. On the other hand, the switching time at which this share is allowed to rise occurs quite early because the outside option of the agent decreases quickly (see Figures 5.b and 5.c).

The following example illustrates the results.

Example 5: We set $T = 1$, $A = 1/2$ and $\beta = 0.8$.

Using the parametric hump-shaped wage profile used to plot Figure 3.a, Figure 5.d illustrates the optimal wage profile for various values of K_0.

[INSERT FIGURES 5a,b,c,d]

4.2 Role of discounting

To show our main result, we have made the simplifying assumption that both the employer and the employee do not discount future earnings. In this section, we relax this assumption and consider the case where both the employer and the employee discount future earnings at a positive rate r.

The problem to be solved by the employer can then be written as follows:

$$\max_{k(\tau)\in[0,1],\ 0\leq w(\tau)} \int_0^T e^{-r\tau}\left[\pi(K(\tau)) + v(1-k(\tau)) - w(\tau)\right] d\tau$$

subject to

$$\dot{K}(\tau) = k(\tau),$$

$$K(0) = 0 \text{ (fixed)},$$

and the non-quitting constraint:

$$\int_t^T e^{-r(\tau-t)} w(\tau)\, d\tau \geq \pi(\beta K(t)) \int_t^T e^{-r(\tau-t)} d\tau \text{ for all } t \in [0,T].$$

It is convenient to rewrite the non-quitting constraint as follows:

$$Z(t) \geq e^{-rt} R(t) \text{ for all } t \in [0,T],$$

where $Z(t) \equiv \int_t^T e^{-r\tau} w(\tau)\, d\tau$ is the sum of present value wages to be paid after time t, and $e^{-rt}R(t) \equiv \int_t^T e^{-r\tau}\pi(\beta K(t))d\tau$ is the present value of the outside option for an employee if she quits at t.

The following proposition shows how our main result is affected when the discount rate is positive:

Proposition 6 [optimal time allocation, $r > 0$]: *When the employer and the employee share the same discount rate $r > 0$, $\pi(K) = AK$ and $v(l) = l - \frac{1}{2}l^2$, the second best solution has the following properties. If $k(t) > 0$ over $[0,T)$:*

(i) There exists at least one time interval (t', t'') over which $k(\tau)$ is increasing, i.e. the employer promises a phase of acceleration of human capital accumulation in order to induce the employee to stay longer.

(ii) There exists $t_b < T$, such that over the interval $[t_b, T]$, $k(t)$ will be falling ($\dot{k} < 0$).

(iii) There exists $t_a \geq 0$, such that over the interval $[0, t_a]$, $k(t)$ will be falling ($\dot{k} < 0$).

(iv) The necessary conditions are sufficient.

This proposition shows that our results are not qualitatively affected by a change in the discount rate. The following numerical example illustrates this point.

Example 6: We set $T = 1$, $A = 1/2$ and $\beta = 0.8$.

[INSERT FIGURES 6a,b,c,d]

Figure 6.a represents the present value of the outside option $e^{-rt}R(t)$ for different values of the discount rate (for a realistic value of 2% and for a very large value of 100%). As in the case without discounting, the outside option increases (in present value), then reaches a plateau and finally decreases. Figures 6.b and 6.c illustrate the share of time devoted to knowledge accumulation and the cumulated knowledge, respectively. There is no qualitative difference when the discount rate varies ($r = 0$, $r = 0.02$, $r = 1$). For a discount rate of 2%, the curve of the share of time devoted to knowledge accumulation is slightly above the corresponding curve in the case of no discounting. Similarly, the curve of cumulated knowledge, K, is only slightly affected by an increase in the discount rate from 0% to 2%. Using the parametric hump-shaped wage profile used to plot Figure 3.a, Figure 6.d represents this optimal wage profile for various values of r.[10]

[10] One may wish to introduce an exogenous (Poisson) quit process with arrival rate $q > 0$ into the model. It turns out that this additional feature interacts with the discount rate r in a non-trivial way. Quitting probability introduces a non-trivial distortion (because of the non-quitting constraint).

4.3 Concave utility

Up to this point, we have considered that the employee's utility function is linear, i.e. that the utility derived from wage $w(t)$ is exactly $w(t)$. Let us relax this assumption and consider the case of a concave utility function $u(w)$, with $u(0) = 0$, $u'(w) > 0$ and $u'' \leq 0$ for all $w \geq 0$. Let $U(t)$ be a new state variable denoting, for the "promised utility" (as in Spear and Srivastava 1987) starting at t, whether the employee stays with the firm until the time horizon T:

$$U(t) = \int_t^T u(w(\tau))\, d\tau.$$

This "promised utility" must exceed what the employee can achieve if she quits at t,

$$U(t) \geq R(t) \equiv (T - t)\, u(\pi(\beta K(t))).$$

Since the wage cannot be negative, $w(\tau) \geq 0$, we have

$$\dot{U}(t) = -u(w(\tau)) \leq 0.$$

The employer's optimization programme is to choose the time path $k(t)$ and $w(t)$ in order to maximize the following expression (we consider the case where $r = 0$ and $K_0 = 0$):

$$\max_{k(\tau)\in[0,1],\ 0\leq w(\tau)} \int_0^T [\pi(K(\tau)) + v(1 - k(\tau)) - w(\tau)]\, d\tau$$

subject to the non quitting constraint:

$$U(t) \geq R(t) \equiv (T - t)\, u(\pi(\beta K(t))) \text{ for all } t \in [0, T], \tag{14}$$

and,

$$\dot{K}(t) = k(t),\ 0 \leq k(t) \leq 1,\ w(t) \geq 0,$$

$$\dot{U}(t) = -u(w(\tau)).$$

Note that $K_0 = 0$ is given and $K(T)$ is free.

The following proposition shows how our main result is affected when the utility is concave (for simplicity, we assume an Inada condition, $v'(1) = 0$).[11]

Proposition 7 [Concave utility, $u(w)$]: *Assume that the employee has a (strictly) concave utility function, $u(0) = 0$, $u' > 0$ and $u'' < 0$. If $k(t)$ is interior over $[0, T)$, there exists at least one time interval (t', t'') over which $k(\tau)$ is increasing, i.e. the employer promises a phase of acceleration of human capital accumulation in order to induce the agent to stay longer.*

[11] It can be shown that the wage profile is not unique even when $u(w)$ is strictly concave. This is because $u(w)$ appears indirectly in an integral constraint, and does not appear directly in the objective function.

5 Link with empirical evidence

Our theoretical results are broadly consistent with the empirical evidence on life-cycle human capital accumulation, and wage profiles.

Human capital accumulation being difficult to measure, the empirical literature provides mainly documentation of formal on-the-job training. The evidence is consistent with our result that human capital accumulation is not concentrated at the beginning of the career.[12]

Indeed, Loewenstein and Spletzer (1997) show that delayed formal training is frequent.[13] More precisely, they show that, for a given job duration, more workers receive training in year $t+1$ than in year t (see Loewenstein and Spletzer 1997, Table 3). This is consistent with our result that k may be increasing, k being interpreted as the probability for a worker to receive training at time t. This is not the case in Ben-Porath's model.

Concerning life-cycle wage profiles, we have shown in subsection 3.2 that our theory can produce a hump-shaped wage profile, in line with Heckman (1976) and Rosen (1976). In particular, a strictly concave wage profile, with the biggest increase in wages occuring at the beginning of a career, as in Fig. 2 in Murphy and Welch (1990), is consistent with our model when $K(0)$ is sufficiently high.

6 Discussion and related literature

As already mentioned, our paper contributes to the literature on the accumulation of human capital within an organizational context. Other papers have considered the human capital accumulation problem within an organization. Smid and Volkerink (1999) extend the analysis of specific schooling by Hashimoto (1981) by introducing non-specific or general schooling within a two-period framework. In the first period, the employer and the employee not only have to choose the level of investment in human capital and the division of costs and benefits, but also have to decide on the specificity of the training. In the second period, (private) information on the productivity of schooling becomes available, and the employee may decide to quit, or the employer may dismiss the employee. They analyze the consequences of subsidies or taxes on schooling. The degree of specificity of training is also an important feature of our paper. However, we consider the specificity of knowledge as an exogenous parameter and analyse its influence on knowledge accumulation and on the

[12] Bartel (1995) finds that 47% of the individuals hired before 1980 have received some formal training by 1990. This empirical result suggests that human capital investments are not only undertaken at the beginning of a career.

[13] They present several lines of evidence. They first show that the probability of job separation is negatively correlated with tenure (using the 1988-1991 National Longitudinal Survey of Youth, NLSY). Then, they focus on the timing of training. They show that the relationship between tenure and the probability of having ever received training is positive (using the 1991 supplement to the Current Population Survey, CSP). They argue that there are several interpretations of this result: either training increases with tenure or those workers who form good matches both receive training and stay longer within firms, whereas bad matches do not receive training and are more likely to terminate. They finally show that, among groups of workers who change jobs after the same number of years, more workers receive training in year t+1 than in year t. They argue that the simple human capital model predicts that training will occur as soon as possible, and that delay in training may be due to delay in the revelation of information about the quality of employer-employee match. Our model is an alternative explanation for this evidence.

life-time earning of the knowledge worker.

As in the present paper, Azariadis (1987) and Bernhard and Timmis (1990) consider models of human capital accumulation problems with contracting. Azariadis (1987) considers implicit contracts as devices for redistributing returns to human capital over time when workers cannot borrow (in an overlapping generation model). He provides a condition for self-enforcing contracts to exist, and explores how existence depends on the rate of interest and on human capital specificity. Bernhard and Timmis (1990) focus on the implications of capital market imperfections for the form of long term wage contracts. As in Azariadis (1987), firms can mediate financially for workers to reallocate returns to human capital over time. Workers can smooth their consumption over time, and human capital accumulates according to an exogenous process. Differently, the present paper focuses on endogenous human capital accumulation.

Our paper contributes to the literature on the hold-up problem[14] by exploring a trade-off that has not been analysed previously. On the one hand, by letting the knowledge worker accumulate knowledge early, the employer obtains large returns on this investment as long as the worker stays with the firm, but the incentives for the worker to quit *become stronger over time* until she approaches her retirement age. On the other hand, letting the worker accumulate knowledge later, the employer obtains lower returns because the remaining horizon is shorter, but the worker's incentives to quit become weaker. Hvide and Kristiansen (2012) also deal with the management of knowledge workers. They study how both firm-specific complementary assets and intellectual property rights affect the management of knowledge workers. They focus on the trade-off between moral hazard and hold-up.

Our model is related to dynamic models of hold-up. Guriev and Kvasov (2005) show that, if contracts are allowed to extend *beyond the breakup* of relationship, efficiency can then be achieved by a sequence of constantly renegotiated fixed term contracts, or by a perpetual contract that allows unilateral termination with advanced notice. In our model, after the breakup the employer is allowed only limited compensation, and therefore efficiency is not achieved. In Pitchford and Snyder (2004) the seller can make gradual investment installments, and the buyer has an incentive to pay after each installment, so the hold-up problem can be mitigated if there is no known, finite end to the number of installments. In our model, there is a finite end to the installments because the time horizon is finite. Furthermore, installment payments are not possible because the employee cannot borrow, and does not have adequate income to post a bond during training.[15]

Che and Sakovics (2004) consider the role of anticipated future investments in a joint project. Two

[14] The hold-up problem is a crucial factor in the determination of the evolution of a bilateral relationship when agents make, ex ante, sunk and specific investments which will increase, ex post, the surplus of the relationship. Then, being unable, ex ante, to secure a share of the surplus in relation to the amount of their sunk investments, later agents will have to negotiate the division of the surplus, taking account that their bargaining power will have changed. This is "the fundamental transformation" (Williamson, 1979), the value of their specific investments being different "out of the relationship" than "within".

[15] Various remedies have been proposed as safeguards against holdup, ranging from vertical integration (Klein et al. 1978, Williamson 1979), property rights allocation (Grossman and Hart, 1986, Hart and Moore, 1990), contracting on renegotiation rights (Chung, 1991, Aghion et al., 1994), option contracts (Nöldeke and Schmidt, 1995, 1998), production contracts (Edlin and Reichelstein, 1996), relational contracts (Baker et al. 2002), financial rights allocation (Aghion and Bolton,1992, Dewatripont and Tirole, 1994, and Dewatripont et al. 2003) and hierarchical authority (Aghion and Tirole 1997) to injecting market competition (MacLeod and Malcomson 1993, Acemoglu and Shimer 1999, Felli and Roberts 2011, and Che and Gale 2003).

partners make investments and negotiate to share the surplus within an infinite horizon setting. Players receive payments only when an agreement is reached (and the game ends). Their main result is that for sufficiently patient players, the hold-up problem may be alleviated because of the shadow of the future. Smirnov and Wait (2004) focus on the timing of investments in a bilateral relationship where (i) two players can invest only once, and (ii) trade occurs only once and only if both players have made a specific investment. Their paper shows that if the potential investment horizon is continually extended, players move in alternation from a prisoners' dilemma to a coordination game. Our model is very different from those of Che and Sakovics (2004) and Smirnov and Wait (2004) as we study a situation where players get payments all along the game. Compte and Jehiel (2003) study the effect of the outside option (the value of which changes over time, depending on the history of offers and concessions) on the equilibrium of a bargaining game. They consider a complete information game that has two features. First, at any point of time, each party has the option to terminate the game (in contrast to our model, where only one party, namely the employee, has this option). Second, in case of termination, the payoffs are assumed to depend on the history of offers or concessions made in the bargaining process. They show that for a large class of such games, gradualism is a necessary feature. They also derive an upper bound on concessions.

Finally, even though our model's main focus is on how the time path of human capital accumulation is distorted because of incomplete contracts, it is also potentially useful for thinking about the "exploration and exploitation" relationship which has been discussed in the management literature where adaptive processes balance between the exploration of new possibilities and the exploitation of old certainties (Schumpeter 1934, Holland 1975). According to March (1991), "exploration includes things captured by terms such as search, variation, risk taking, experimentation, play, flexibility, discovery, innovation", while exploitation "includes such things as refinement, choice, production, efficiency, selection, implementation, execution". As noted by Ben Porath (1967), the pioneering work of Becker (1962, 1964) emphasized the demand side of human capital formation. Thus, Becker's focus is on the exploitation motive. Ben Porath (1967) emphasizes the supply side, the investment costs. This reflects the exploration motive. In our paper we consider the two interrelated motives within an organization where the value of the outside option evolves endogenously. Our non-monotonicity result, which shows that the rate of change in knowledge comes in sudden bursts, may be viewed in the light of the theory of punctuated dynamics (Eldredge and Gould 1972), according to which evolution is not a gradual process. Periods of exploration where things change drastically are inserted between prolonged periods of exploitation where only minor changes take place.

7 Conclusion

In this paper, we have considered a knowledge accumulation problem within an organization. We have shown how the employer optimally distorts the knowledge accumulation path and chooses a wage profile in order to mitigate the opportunism problem. We have shown that knowledge accumulation is delayed. The fraction of working time allocated to knowledge creation is highest at the early career stage, falls gradually, then rises again, before falling finally toward zero. We have also shown that an increase in the severity of the enforcement problem (or a decrease in the degree of firm-specificity of knowledge) decreases the share of working time allocated to knowledge creation, and decreases cumulated knowledge. Moreover, such an increase can increase or decrease the life-time income of the knowledge worker. Our main result is robust to an increase in the initial knowledge level, to a change in the discount rate, and to a change in the worker's utility function.

Our model fits well with existing empirical evidence. Our main result, the non-monotonicity of investment in human capital, fits well with the reported evidence that delayed formal training (i.e. an increasing quantity of formal training over time) is frequent. Our model matches also the main feature of age-earning profiles, that the optimal experience-earning profile is a hump-shaped function of experience (among other solutions). And, consistently with recent evidence that differences in lifetime earnings are mainly due to initial conditions, we have found that an increase in the initial knowledge level has a positive effect on earnings.

Appendix A: Proofs

Proof of Proposition 1: Since $\widehat{\pi}(K) < \pi(K)$, the value of knowledge is larger when the employee stays with the firm, the optimal quitting time is $t^* = T$. The joint surplus maximization problem reduces to

$$\max_k \int_0^T e^{-r\tau} \left[\pi\left(K\left(\tau\right)\right) + v\left(1 - k\left(\tau\right)\right)\right] d\tau,$$

where $0 \le k(\tau) \le 1$ and $\dot{K}(\tau) = k(\tau)$.

Let ψ be the co-state variable, λ and μ be the multipliers associated with the constraint $k \ge 0$ and $1 - k \ge 0$. Write the Lagrangian

$$\mathcal{L} = e^{-r\tau}\pi(K) + e^{-r\tau}v(1-k) + \psi k + \lambda k + \mu(1-k)$$

The necessary conditions are

$$\frac{\partial \mathcal{L}}{\partial k} = -e^{-r\tau}v'(1-k) + \psi + \lambda - \mu = 0 \tag{15}$$

$$\lambda \ge 0, \qquad k \ge 0, \qquad 1 \ge k \qquad \lambda k = 0, \qquad \mu(1-k) = 0 \tag{16}$$

$$-\dot{\psi} = \frac{\partial \mathcal{L}}{\partial K} = e^{-r\tau}\pi'(K) \tag{17}$$

and the transversality condition is

$$\psi(T) = 0 \tag{18}$$

Integration of (17) gives $-\left[\psi(T) - \psi(t)\right] = \int_t^T \pi'\left(K\left(\tau\right)\right) e^{-r\tau} d\tau$. It follows that

$$\psi(t) = \int_t^T \pi'\left(K\left(\tau\right)\right) e^{-r\tau} d\tau > 0 \text{ for all } t < T \tag{19}$$

Using (17), we have

$$\dot{\psi} = -e^{-r\tau}\pi'(K) < 0. \tag{20}$$

Assuming an interior solution, we have $\mu(t) = \lambda(t) = 0$, and using (15), we have $\psi = e^{-r\tau}v'(1-k)$; and using (19), we have $v'(1-k) = \int_t^T \pi'(K(\tau))\, e^{-r(\tau-t)}d\tau$. Differentiating, we obtain:

$$\dot{k} = \frac{-\left[\pi'(K) - rv'\right]}{-v''(1-k)}.$$

To show that $\dot{k} < 0$ we must prove that $\pi' > rv'$. Define the current-value shadow price $p = e^{rt}\psi \geq 0$; then $\dot{p} = rp - \pi'$ and $p(T) = 0$. Along the optimal path, rp cannot exceed π' (if it does, then p cannot go to zero). From its definition, $p = v'$ for all $t \in (0,T)$. Therefore $\pi'(K) - rv' > 0$. Thus $\dot{k} < 0$.■

Proof of Proposition 2: Let us show (i), (ii), and (iii).

Let us first show that $\dot{k}(t) > 0$ when $\varphi(t) > 0$ and $\dot{k}(t) < 0$ when $\varphi(t) = 0$.

Assume that $\varphi(t) > 0$ for all $t \in (t', t'')$. Using (7), we have $W(t) = R(t)$ for all $t \in (t', t'')$, or,

$$\int_t^T w(\tau)\, d\tau = (T-t)\,\pi(\beta K(t)).$$

Differentiating this condition, we find:

$$-w(t) = -\pi(\beta K(t)) + (T-t)\,\beta\pi'(\beta K(t))\, k(t). \tag{21}$$

Using $\varphi(t) > 0$ and (9) we have $\dot{\rho} < 0$. Using (5) and (6), we obtain $\rho = \alpha \geq 0$. Since $\dot{\rho} < 0$, we have $\rho = \alpha > 0$ for all $t \in (t', t'')$ and then $w(t) = 0$. Hence, differentiating (21) and rearranging we find:

$$\dot{k}(t) = \frac{2\pi'(\beta K)\, k + \beta(T-t)\left(-\pi''(\beta K)\right) k^2}{(T-t)\,\pi'(\beta K)} > 0. \tag{22}$$

Now assume that $\varphi(t) = 0$ for all $t \in [t_1, t_2]$. Using (8) and differentiating (4), we have $-\pi'(K) = \dot{\psi} = -v''(1-k)\dot{k}$, or,

$$\dot{k} = \frac{-\pi'(K)}{-v''(1-k)} < 0. \tag{23}$$

Let us look for solution such that ψ and ρ are continuous.

Using (4), we have $\psi = v'(1-k) > 0$ and differentiating, we have $\dot{\psi} = -v''(1-k)\dot{k}$. Since $\psi(T) = 0$, there exists an interval $[t_b, T]$ such that $\dot{\psi}(t) < 0$ and then $\dot{k}(t) < 0$ for all $t \in [t_b, T]$.

Assume that there exists $t_1 > 0$ such that $\dot{k}(t) > 0$ for all $t \in [0, t_1]$. Hence, we know from the reasoning above that

$$0 = -w(0) = T\beta\pi'(0)\, k(0),$$

and then $k(0) = 0$. But we have assumed that $k(t) > 0$ for all $t \in [0, T)$. Hence there exists $t_a > 0$ such that $\dot{k}(t) < 0$ for all $t \in [0, t_a]$.

Now assume that there is no phase $\dot{k}(t) > 0$, i.e. we have $\varphi(t) = 0 = \dot{\rho}(t)$ for all t. Since $\rho(0) = 1$, we must have $\rho(t) = 1$ for all t. Using (5), we have $\alpha(t) = \rho(t) = 1 > 0$. Hence $w(t) = 0$ for all t and then

$W(t) = 0$ for all t. Using (7), we have $0 = W(t) \geq R(t)$ and then $R(t) = (T-t)\,\pi(\beta K(t)) = 0$ for all t, i.e. $K(t) = 0$ for all t. But we have assumed that $k(t) > 0$ for all $t \in [0, T)$.

See Appendix C for a proof of (iv). ■

Proof of Proposition 3: Notice that $\dot{R}^*(t_b) = 0$, $\dot{R}^*(t) \leq 0$. Moreover, we have $\ddot{R}^*(t) \leq 0$ over $[t_b, T]$. Indeed,

$$\ddot{R}^*(t) = -2\beta\pi'(\beta K^*)\,k^* + (T-t)\beta^2\pi''(\beta K^*)\,(k^*)^2 + (T-t)\beta\pi'(\beta K^*)\dot{k}^*.$$

Since $\dot{k}^* < 0$ for $t \in (t_b, T)$, we have $\ddot{R}^*(t) < 0$ over (t_b, T).

We are looking for $w(.)$ that satisfies the necessary conditions.

The first condition is the non quitting constraint:

$$W(t) \equiv \int_t^T w(\tau)\,d\tau \geq R^*(t) = (T-t)\,\pi(\beta K^*(t)), \tag{24}$$

The second condition ensures that the principal leaves minimal rents to the agent:

$$W(t_b) = R^*(t_b) \text{ and } W(T) = R^*(T) = 0 \tag{25}$$

In order that the wage profile fits with the evidence, we want to find a hump-shaped wage profile, i.e. we require that:

$$\dot{w}(T) < 0 \leq \dot{w}(t_b) \tag{26}$$

We also need that the wage is non negative:

$$w(t) \geq 0 \text{ for all } t. \tag{27}$$

Let us define the following function:

$$W(t) = \begin{cases} R^*(t_b) \text{ for } t \in [0, t_b] \\ s(t)\,R^*(t_b) + (1-s(t))\,R^*(t) \text{ for } t \in [t_b, T] \end{cases}, \tag{28}$$

where $s(t)$ for all $t \in [0, T]$.

Remark that condition (24) holds because R^* increases over $[0, t_a]$, reaches a plateau over $[t_a, t_b]$, and decreases over $[t_b, T]$. Condition (25) also holds only if

$$s(T) = 0. \tag{29}$$

Using $w = -\dot{W}$ (by definition of W), condition (27) is equivalent to:

$$\dot{W}(t) = \dot{s}(t)\,(R^*(t_b) - R^*(t)) + (1-s(t))\,\dot{R}^*(t) \leq 0 \tag{30}$$

for $t \in [t_b, T]$. Using $\dot{w} = -\ddot{W}$, condition (26) can be rewritten as follows:

$$\ddot{W}(t_b) \leq 0 < \ddot{W}(T). \tag{31}$$

Since

$$\ddot{W}(t) = \ddot{s}(t)\left(R^*(t_b) - R^*(t)\right) - 2\dot{s}(t)\dot{R}^*(t) + (1 - s(t))\ddot{R}^*(t),$$

condition (31) can be rewritten as:

$$\ddot{W}(t_b) = -2\dot{s}(t_b)\dot{R}^*(t_b) + (1 - s(t_b))\ddot{R}^*(t_b) < 0, \tag{32}$$

and, using (29):

$$\ddot{W}(T) = \ddot{s}(T)\left(R^*(t_b) - R^*(T)\right) - 2\dot{s}(T)\dot{R}^*(T) + \ddot{R}^*(T) > 0 \tag{33}$$

Since $\dot{R}^*(t_b) = 0$ and $\ddot{R}^*(t_b) \leq 0$, (32) always holds.

Hence, we have to find $s()$ such that (29), (30) and (33) hold and $s(t) \in [0,1]$ for all t in $[t_b, T]$. Let us consider the following function:

$$s(t) = \frac{\sigma(T-t)e^{-B(t-T)}}{1 + \sigma(T-t)e^{-B(t-T)}}, \tag{34}$$

where $\sigma > 0$ and

$$B = \lambda\left(\frac{-2\sigma\dot{R}^*(T) - \ddot{R}^*(T)}{2\sigma R^*(t_b)} + \sigma\right), \tag{35}$$

where $\lambda > 1$.

First, notice that (29) holds, i.e. $s(T) = 0$. Second, (30) holds because we know that $\dot{R}^*(t) \leq 0$ and $R^*(t_b) \geq R^*(t)$ and

$$\dot{s}(t) = -\sigma\frac{e^{B(T-t)}\left(B(T-t)+1\right)}{\left(\sigma(T-t)e^{B(T-t)}+1\right)^2} < 0. \tag{36}$$

Third, using

$$\ddot{s}(t) = -\sigma\frac{e^{B(T-t)}\left(\sigma\left((B(T-t))^2 + 2(B(T-t)+1)\right)e^{B(T-t)} - 2B - B^2(T-t)\right)}{\left(\sigma(T-t)e^{B(T-t)}+1\right)^3},$$

and, $\dot{s}(T) = -\sigma$ and $\ddot{s}(T) = 2\sigma(B - \sigma)$. Condition (33) writes:

$$\ddot{W}(T) = 2\sigma(B-\sigma)\left(R^*(t_b) - R^*(T)\right) + 2\sigma\dot{R}^*(T) + \ddot{R}^*(T) > 0,$$

or,

$$B > \frac{-2\sigma\dot{R}^*(T) - \ddot{R}^*(T)}{2\sigma\left(R^*(t_b) - R^*(T)\right)} + \sigma,$$

which is true because $\lambda > 1$ in (35). ■

Proof of Proposition 4: We first show point (i). We know from appendix B (taking $K_0 = 0$) that t_a and t_b are the solution of

$$\beta = -\ln\left(\frac{T-t_b}{T-t_a}\right) + \frac{2}{3}\left(1 - \left(\frac{T-t_b}{T-t_a}\right)^3\right) \tag{37}$$

$$\frac{(T-t_b)^3}{T-t_a} = \frac{1}{2}\frac{(T-t_a)(t_a)^2}{(T-2t_a)} \tag{38}$$

Multiplying both sides of (38) by $1/(T-t_a)^2$, we have:

$$\left(\frac{T-t_b}{T-t_a}\right)^3 = \frac{1}{2}\frac{(t_a)^2}{(T-2t_a)(T-t_a)}$$

Plugging this condition into (37), we obtain:

$$\beta = -\frac{1}{3}\ln\left(\frac{1}{2}\frac{(t_a)^2}{(T-2t_a)(T-t_a)}\right) + \frac{2}{3}\left(1-\frac{1}{2}\frac{(t_a)^2}{(T-2t_a)(T-t_a)}\right).$$

This condition characterizes t_a as a function of β. Differentiating this condition with respect to β, we have:

$$\frac{dt_a}{d\beta} = -\frac{3t_a(T-t_a)^2(T-2t_a)^2}{\left((T-2t_a)(T-t_a)+(t_a)^2\right)(2T-3t_a)T} < 0.$$

The function $t_a \mapsto \frac{1}{2}\frac{(t_a)^2(T-t_a)^2}{(T-2t_a)}$ increases with respect to t_a. Indeed, the derivative of the function $t_a \mapsto (t_a)^2(T-t_a)^2$ is given by $t_a \mapsto 4t_a(T-t_a)\left(\frac{T}{2}-t_a\right) > 0$. Using (38), we have that t_b decreases when t_a increases. Hence, t_b increases with respect to β.

Now consider point (ii). We know from the examples that the optimal cumulated knowledge is given by

$$K(t) = \begin{cases} \left(A\frac{(T-t_b)^3}{(T-t_a)^2} + At_a - A\frac{t}{2}\right)t \text{ if } t\in[0,t_a] \\ A\frac{(T-t_b)^3}{T-t} \text{ if } t\in[t_a,t_b] \\ A(T-t_b)^2 + A\left((t-t_b)T - \frac{1}{2}t^2 + \frac{1}{2}t_b^2\right) \text{ if } t\in[t_b,T] \end{cases}, \tag{39}$$

and then,

$$\frac{dK(t)}{d\beta} = \begin{cases} \left(-3\frac{A(T-t_b)^2}{(T-t_a)^2}\frac{dt_b}{d\beta} + \left(2\frac{A(T-t_b)^3}{(T-t_a)^3}+A\right)\frac{dt_a}{d\beta}\right)t < 0 \text{ if } t\in[0,t_a] \\ -3A(T-t_b)^2\frac{1}{T-t}\frac{dt_b}{d\beta} < 0 \text{ if } t\in[t_a,t_b] \\ -2A(T-t_b)\frac{dt_b}{d\beta} < 0 \text{ if } t\in[t_b,T] \end{cases}.$$

The optimal share of time devoted to knowledge accumulation is given by

$$k(t) = \begin{cases} A\frac{(T-t_b)^3}{(T-t_a)^2} + At_a - At \text{ for } t\in[0,t_a], \\ A(T-t_b)^3(T-t)^{-2} \text{ for } t\in[t_a,t_b], \\ A(T-t) \text{ for } t\in[t_b,T], \end{cases} \tag{40}$$

and then

$$\frac{dk(t)}{d\beta} = \begin{cases} -3\frac{A(T-t_b)^2}{(T-t_a)^2}\frac{dt_b}{d\beta} + \left(2\frac{A(T-t_b)^3}{(T-t_a)^3}+A\right)\frac{dt_a}{d\beta} < 0 \text{ for } t\in[0,t_a], \\ -A(T-t_b)^2(T-t)^{-2}\frac{dt_b}{d\beta} < 0 \text{ for } t\in[t_a,t_b], \\ 0 \text{ for } t\in[t_b,T], \end{cases}$$

Now consider point (iii). Using (39) for $t=T$, the total amount of accumulated knowledge is given by $K(T) = \frac{3}{2}A(T-t_b)^2$. Hence $K(T)$ decreases with respect to β.

We also have $W_0 = R(t_b) = (T-t_b)\beta AK(t_b) = \beta A^2(T-t_b)^3$. ■

Proof of Proposition 5:

We first prove points (i), (ii) and (iv). We can easily show that the derivative of k with respect to K_0 is

$$\frac{dk(t)}{dK_0} = \begin{cases} -3\Delta^2 A\frac{d(t_b-t_a)}{dK_0} + A(1-\Delta)\left(1+\Delta-2\Delta^2\right)\frac{dt_a}{dK_0} < 0 \text{ for } t\in[0,t_a], \\ -A(T-t_b)^2(T-t)^{-2}\frac{dt_b}{d\beta} > 0 \text{ for } t\in[t_a,t_b], \\ 0 \text{ for } t\in[t_b,T], \end{cases}$$

and the derivative of the cumulative knowledge is (using the fact that Δ does not depend on K_0):

$$\frac{dK(t)}{dK_0} = \begin{cases} 1 + A\left(1-\Delta^3\right)\frac{dt_a}{dK_0}t > 0 \text{ if } t \in [0, t_a] \\ -3A\left(T-t_b\right)^2 \frac{1}{T-t}\frac{dt_b}{dK_0} > 0 \text{ if } t \in [t_a, t_b] \\ -2A\left(T-t_b\right)\frac{dt_b}{dK_0} > 0 \text{ if } t \in [t_b, T] \end{cases}$$

The total wage is given by $W_0 = R(t_b) = A\beta(T-t_b)K(t_b) = A^2\beta(T-t_b)^3$, and then, its derivative with respect to K_0 is

$$\frac{dW_0}{dK_0} = -3A^2\beta(T-t_b)^2\frac{dt_b}{dK_0} > 0.$$

We now prove point (iii). From appendix B, we have:

$$A\Delta^3 = \frac{2K_0 + A(t_a)^2}{2(T-2t_a)(T-t_a)}, \tag{41}$$

and $\Delta \in (0,1)$ is the solution of

$$\beta = -\ln(\Delta) + \frac{2}{3}\left(1-\Delta^3\right), \tag{42}$$

with $\Delta = \frac{T-t_b}{T-t_a}$.

Using (42) reveals that Δ does not depend on K_0. Using (41) and the fact that $t_a \mapsto \frac{2K_0+A(t_a)^2}{2(T-2t_a)(T-t_a)}$ is an increasing function, we deduce that an increase of K_0 leads to an decrease of t_a. Since Δ is not affected by the increase of K_0, t_b also decreases as K_0 increases (and then $\frac{dt_b}{dK_0} = -\Delta\frac{dt_a}{dK_0}$). Also, since $\Delta = \frac{T-t_b}{T-t_a}$, we have $t_b - t_a = \frac{1-\Delta}{\Delta}(T-t_b)$ and then $t_b - t_a$ increases when K_0 increases. ■

Proof of Proposition 6: The firm offers to the employee a package deal whereby it pays her a sum $w(\tau)$ at time τ, provided that she stays with the firm until time τ for all $\tau \in [0,T]$, and works for the firm according to a prescribed time allocation scheme $(k(\tau), 1-k(\tau))$ where $k(\tau)$ is the fraction of the workday to be spent to knowledge creation, and $1-k(\tau)$ is the fraction of the day to be spent on other tasks. If the employee quits at any time $t < T$, her full income stream from time t to T will be

$$R(t) = \int_t^T e^{-r(\tau-t)}\pi(\beta K(t))d\tau = \pi(\beta K(t))e^{rt}\theta(t) \tag{43}$$

where $\theta(t) \equiv \frac{e^{-rt}-e^{-rT}}{r}$. Remark that, as $r \to 0$, the RHS goes to $(T-t)\pi(\beta K(t))$.

The employee will stay with the firm until T if and only if $V_w(t) \geq R(t)$ for all $t \in [0,T]$. Thus, the firm's package deal must satisfy the non-quitting constraint

$$\int_t^T e^{-r(\tau-t)}w(\tau)\,d\tau \geq \pi(\beta K(t))e^{rt}\theta(t) \quad \text{for all } t \in [0,T],$$

or,

$$\int_t^T e^{-r\tau}w(\tau)\,d\tau \geq \pi(\beta K(t))\theta(t) \quad \text{for all } t \in [0,T]. \tag{44}$$

Let us define the remaining wage (discounted from 0) as a new state variable $Z(t)$:

$$Z(t) \equiv Z_0 - \int_0^t z(\tau)d\tau \text{ for all } t \in [0,T], \tag{45}$$

where Z_0 is the present value of all future wage payments, and z is the discounted wage: $z(t) \equiv e^{-rt}w(t)$ for all $t \in [0,T]$. The non-quitting constraint (44) can then be rewritten as:

$$Z(t) \geq \pi(\beta K(t))\theta(t) \quad \text{for all } t \in [0,T], \tag{46}$$

and the dynamics of Z is given by $\dot{Z}(t) = -z(t)$ for all $t \in [0,T]$.

The firm must now choose the time path of the control variables $k(t)$ and $z(t)$ to maximize the objective function

$$\begin{aligned} V_f &= \int_0^T e^{-r\tau}\left[\pi(K(\tau)) + v(1-k(\tau)) - w(\tau)\right] d\tau \\ &= \int_0^T e^{-r\tau}\left[\pi(K(\tau)) + v(1-k(\tau))\right] d\tau - Z_0, \end{aligned} \tag{47}$$

subject to

$$\begin{aligned} \dot{K} &= k \\ K(0) &= 0 \end{aligned} \tag{48}$$

$$\dot{Z} = -z$$

$$0 \leq k(t) \leq 1,$$

$$z(t) \geq 0.$$

and the non quitting constraint, using (43) and (46),

$$Z(t) \geq R(t)e^{-rt} \equiv \pi(\beta K(t))\theta(t) \tag{49}$$

Note that Z_0 must be chosen, which implies a cost, as seen in (47). There are no explicit restrictions on $K(T)$ and $Z(T)$.

Necessary conditions: Let $\psi(t)$ and $\rho(t)$ be the co-state variables associated with the state variables $K(t)$ and $Z(t)$ respectively. Let $\lambda(t)$, $\mu(t)$, $\alpha(t)$ and $\varphi(t)$ be the multipliers associated with the inequality constraints $k(t) \geq 0$, $1-k(t) \geq 0$, $z(t) \geq 0$ and $Z(t) \geq R(t)e^{-rt} = \pi(\beta K(t))\theta(t)$. Define the Hamiltonian H and the Lagrangian $\pounds$ as follows

$$H = e^{-rt}\pi(K) + e^{-rt}v(1-k) + \psi k - \rho z(t),$$

$$\pounds = H + \lambda k + \mu[1-k] + \alpha z + \varphi\left[Z - Re^{-rt}\right].$$

The necessary conditions include

$$\frac{\partial \pounds}{\partial k} = -e^{-rt}v'(1-k) + \psi + \lambda - \mu = 0, \tag{50}$$

$$\frac{\partial \pounds}{\partial z} = -\rho + \alpha = 0, \tag{51}$$

$$\alpha \geq 0, \lambda \geq 0,\ k \geq 0,\ z \geq 0,\ \lambda k = 0,\ \mu \geq 0\,,\ 1-k \geq 0,\ \mu(1-k) = 0, \alpha z = 0 \tag{52}$$

$$\varphi \geq 0,\ Z - R \geq 0,\ \varphi\left(Z - Re^{-rt}\right) = 0 \tag{53}$$

$$-\dot{\psi} = \frac{\partial \pounds}{\partial K} = e^{-rt}\pi'(K) - \varphi\left(\beta\theta(t)\,\pi'\right) \tag{54}$$

$$-\dot{\rho} = \frac{\partial \pounds}{\partial Z} = \varphi \tag{55}$$

The following transversality conditions are also necessary. Since Z_0 is to be chosen optimally, at a cost, and there are no restrictions on $K(T)$ and $Z(T)$, we have

$$\psi(T) = 0 \tag{56}$$

$$\begin{aligned} \rho(0) &= 1-\delta,\ \delta\left(Z(0) - R(0)\right) = 0, \\ \delta &\geq 0 \text{ and } Z(0) - R(0) \geq 0, \end{aligned} \tag{57}$$

$$\rho(T) = 0. \tag{58}$$

We look at a solution such that k is interior and the total wage $Z(0) > 0$.

Using $R(0) = A\beta K(0)\left[\frac{1-e^{-rT}}{r}\right] = 0$ (from (48)), and (57), we have $\delta = 0$ and then

$$\rho(0) = 1 \tag{59}$$

Using our specification of $\pi(K) = AK$, $v(l) = l - \frac{1}{2}l^2$, conditions (50), (51), (54) and (55) can be rewritten as follows:

$$\psi = e^{-rt}k, \tag{60}$$

$$\rho = \alpha \geq 0, \tag{61}$$

$$\dot{\psi} = A\left[\varphi\beta\theta - e^{-rt}\right], \tag{62}$$

$$\dot{\rho} = -\varphi. \tag{63}$$

From (59), (58) and (63),

$$\int_0^T \varphi(t)dt = 1. \tag{64}$$

This implies that there exists some interval $[t_a, t_b]$ in which the constraint $Z(t) - e^{-rt}R(t)$ is binding.

Remark: *Also, notice that if we write $S(t) \equiv e^{-rt}R(t)$, then $S(t) = A\beta K(t)\theta(t)$ and $S(0) = S(T) = 0$. We also have $\dot{S} = A\beta\left(k\theta + K\dot{\theta}\right)$. Thus $\dot{S}(0) = A\beta k(0)\,\theta(0) > 0 > \dot{S}(T) = -A\beta K(T)\,e^{-rT}$.*

Let us find a solution such that ρ is continuous and $\rho(t) = 1$ for $t \in [0, t_a)$, $\dot{\rho}(t) < 0$ for $t \in [t_a, t_b]$ and $\rho(t) = 0$ for $t \in (t_b, T]$. We then have $\rho(t) > 0$ for $t \in [0, t_b)$.

First consider $t \in [0, t_a)$. We have $\dot{\rho}(t) = 0$. Then (63) implies that $\varphi(t) = 0$. Using (62) and differentiating (60), we find $-Ae^{-rt} = \dot{\psi} = -re^{-rt}k + e^{-rt}\dot{k}$, or, $-A = \dot{k} - rk$. Letting $k(0) = k_0$ (to be determined later), we find

$$k(t) = k_0 e^{rt} + \frac{A}{r}\left(1 - e^{rt}\right) \text{ for all } t \in [0, t_a), \tag{65}$$

which implies that $\dot{k}(t) < 0$ over $[0, t_a)$ iff $A - rk_0 > 0$. Note that as $r \to 0$, the RHS tends to $k_0 - At$.

Hence,

$$K(t) = K_0 + \left(k_0 - \frac{A}{r}\right)\left(\frac{e^{rt} - 1}{r}\right) + \frac{A}{r}t \text{ for all } t \in [0, t_a). \tag{66}$$

Second, consider $t \in (t_a, t_b)$. Using $\dot{\rho}(t) < 0$ and (63) we have $\varphi(t) > 0$ and then (53) implies that

$$Z(t) = S(t) \text{ for all } t \in (t_a, t_b). \tag{67}$$

Moreover, since $\rho(t) > 0$ for $t \in [0, t_b)$, (51) and (52) imply that $z(t) = 0$ and $Z(t) = 0$ for all $t \in [0, t_b)$. Hence, (67) implies that $S = A\beta K\theta$ is constant over (t_a, t_b). Then

$$K(t) = \frac{D}{\theta(t)} \text{ for all } t \in (t_a, t_b), \tag{68}$$

where D is a constant to be determined latter.

Differentiating (68), we find:

$$k(t) = -D\frac{\dot{\theta}(t)}{(\theta(t))^2} \text{ for all } t \in (t_a, t_b), \tag{69}$$

and differentiating again, we have:

$$\dot{k} = D\frac{2\theta\left(\dot{\theta}\right)^2 - \ddot{\theta}\theta^2}{\theta^4}, \tag{70}$$

or,

$$\dot{k} = r^3 D e^{-rt}\frac{e^{-rt} + e^{-rT}}{\left(e^{-rt} - e^{-rT}\right)^3} > 0.$$

Third, consider $t \in (t_b, T]$. Since $\rho(0) = 0$ for $t \in (t_b, T]$, we have $\dot{\rho}(t) = 0$ for $t \in (t_b, T]$. Hence, (63) implies that $\varphi(t) = 0$. Using (62) and differentiating (60), we find $-A = \dot{k} - rk$. Using (56) and (60), we find $k(T) = 0$. Hence,

$$k(t) = \frac{A}{r}\left(1 - e^{-r(T-t)}\right) \text{ for all } t \in (t_b, T]. \tag{71}$$

Since the state variable K is continuous at all t, hence also at $t = t_b$, integrating this equation leads to

$$K(t) = \frac{D}{\theta(t_b)} + \frac{A}{r}(t - t_b) - \frac{A}{r^2}\left(e^{-r(T-t)} - e^{-r(T-t_b)}\right) \text{ for all } t \in (t_b, T]. \tag{72}$$

Notice that we found

$$k(t) = \begin{cases} k_0 e^{rt} + \frac{A}{r}\left(1 - e^{rt}\right) \text{ for } t \in [0, t_a) \\ r^2 D\frac{e^{-rt}}{\left(e^{-rt} - e^{-rT}\right)^2} \text{ for } t \in [t_a, t_b] \\ \frac{A}{r}\left(1 - e^{-r(T-t)}\right) \text{ for } t \in (t_b, T] \end{cases}$$

and,

$$K(t)=\begin{cases} K_0+\left(k_0-\frac{A}{r}\right)\left(\frac{e^{rt}-1}{r}\right)+\frac{A}{r}t \text{ for } t\in[0,t_a) \\ rD\frac{1}{e^{-rt}-e^{-rT}} \text{ for } t\in[t_a,t_b] \\ \frac{rD}{e^{-rt_b}-e^{-rT}}+\frac{A}{r}(t-t_b)-\frac{A}{r^2}\left(e^{-r(T-t)}-e^{-r(T-t_b)}\right) \text{ for } t\in(t_b,T] \end{cases}$$

Solving for t_a, t_b, k_0 and D:

We need four additional conditions to determine the values of t_a, t_b, k_0 and D.

Since the state variable K is continuous at all t, hence at $t=t_a$, this implies:

$$K_0+\left(k_0-\frac{A}{r}\right)\left(\frac{e^{rt_a}-1}{r}\right)+\frac{A}{r}t_a = \frac{D}{\theta(t_a)}. \tag{73}$$

Assuming that k is continuous at $t=t_a$, we have:

$$k_0e^{rt_a}+\frac{A}{r}\left(1-e^{rt_a}\right)=r^2D\frac{e^{-rt_a}}{\left(e^{-rt_a}-e^{-rT}\right)^2} \tag{74}$$

Assuming that k is continuous at $t=t_b$, we have:

$$-D\frac{\dot{\theta}(t_b)}{(\theta(t_b))^2}=\frac{A}{r}\left(1-e^{-r(T-t_b)}\right). \tag{75}$$

We need a fourth condition: using (60), (62) and (63), we find that, for all $t\in(t_a,t_b)$:

$$\dot{\rho}(t)=-\varphi(t)=\frac{1}{A\beta}\frac{\dot{\theta}(t)}{\theta(t)}\left(A+\dot{k}(t)-rk(t)\right).$$

Using (70) and (69), and using $\ddot{\theta}=re^{-rt}=-r\dot{\theta}$, we have

$$\begin{aligned} \dot{\rho}(t) &= \frac{1}{A\beta}\frac{\dot{\theta}}{\theta}\left(A+2D\frac{\dot{\theta}}{\theta^2}\left(\frac{-re^{-rT}}{e^{-rt}-e^{-rT}}\right)\right) \\ &= \frac{1}{A\beta}\frac{\dot{\theta}}{\theta}\left(A-2De^{-rT}\frac{\dot{\theta}}{\theta^3}\right) \\ &= \frac{1}{\beta}\frac{\dot{\theta}}{\theta}-2\frac{D}{A\beta}e^{-rT}\frac{\left(\dot{\theta}\right)^2}{\theta^4}. \end{aligned} \tag{76}$$

Using $\dot{\rho}(t)=0$ for all $t\in[0,t_a)$, we have $\rho(t_a)=\rho(0)=1$. Using, $\rho(t_b)=0$ and integrating (76) between t_a and t_b, we find:

$$\begin{aligned} \rho(t_b)-\rho(t_a) &= -1 \\ &= \int_{t_a}^{t_b}\left(\frac{1}{\beta}\frac{\dot{\theta}}{\theta}-2\frac{D}{A\beta}e^{-rT}\frac{\left(\dot{\theta}\right)^2}{\theta^4}\right)d\tau. \end{aligned}$$

Using integration by parts, this condition can be written as follows:

$$-1=\frac{1}{\beta}\ln\left(\frac{\theta(t_b)}{\theta(t_a)}\right)-\frac{r^3D}{3A\beta}e^{-rT}\left[\frac{3e^{-rt_b}-e^{-rT}}{\left(e^{-rt_b}-e^{-rT}\right)^3}-\frac{3e^{-rt_a}-e^{-rT}}{\left(e^{-rt_a}-e^{-rT}\right)^3}\right]. \tag{77}$$

Solving for (73), (74), (75) and (77) provides the values of t_a, t_b, k_0 and D.

Finally, we must show that $\dot{k}(t) < 0$ over $[0, t_a)$. Combining (74) and (75), we find:

$$k_0 - \frac{A}{r} = -\frac{A}{r}\left[\left(\frac{\theta(t_b)}{\dot{\theta}(t_b)}\frac{\dot{\theta}(t_a)}{\theta(t_a)}\right)^2 \theta(t_b) + \theta(t_a)\right] < 0.$$

Hence, using (65), we conclude that $\dot{k}(t) < 0$ over $[0, t_a)$.

The proof of (iv) is presented in Appendix C. ■

Proof of Proposition 7:

The derivative of R with respect to t, evaluated at $t = 0$, is given by:

$$\lim_{t\to 0} \dot{R}(t) = u'(\pi(\beta K_0))\,\pi'(\beta K_0)\,\beta k(0)\,T - u(\pi(\beta K_0)).$$

Since $K_0 = 0$, k is interior and $\pi(0) = u(0) = 0$, we have:

$$\lim_{t\to 0} \dot{R}(t) = u'(0)\,\pi'(0)\,\beta k(0)\,T > 0.$$

Hence, the first time $R(t)$ attains its maximum value is at some time $t_a > 0$:

$$R(t) < R(t_a) \text{ for all } t < t_a,$$

and,

$$R(t_a) \geq R(t) \text{ for all } t \in [t_a, T].$$

Then, profit maximization implies that the employer sets $U(t_a) = R(t_a)$, because there is no gain to promise the worker higher utility than his maximal outside option. So, in the optimal program, the constraint (14) holds with equality at t_a. Since $\dot{U} = -u(w) \leq 0$, it follows that, for all $t < t_a$,

$$U(t) \geq U(t_a) = R(t_a) > R(t) \text{ for all } t \in [0, t_a). \tag{78}$$

Then the constraint (14) is not binding for all $t < t_a$.

Let $\psi(t)$ and $\rho(t)$ be the co-state variables associated with the state variables $K(t)$ and $U(t)$ respectively. Let $\alpha(t)$ and $\varphi(t)$ be the multipliers associated with the inequality constraints $w(t) \geq 0$ and $U(t) \geq R(t)$ (we ignore the constraint $0 \leq k(t) \leq 1$ because k is assumed to be interior). Define the Hamiltonian H and the Lagrangian $\pounds$ as follows

$$H = \pi(K) + v(1-k) - w + \psi k - \rho u(w),$$

$$\pounds = H + \alpha w + \varphi[U - R].$$

The necessary conditions include

$$\frac{\partial \pounds}{\partial k} = -v'(1-k) + \psi = 0, \tag{79}$$

$$\frac{\partial \pounds}{\partial w} = -1 - \rho u'(w) + \alpha = 0, \tag{80}$$

$$\alpha \geq 0, w \geq 0, \ \alpha w = 0 \tag{81}$$

$$\varphi \geq 0,\, U - R \geq 0,\, \varphi\left(U - R\right) = 0 \tag{82}$$

$$-\dot{\psi} = \frac{\partial \pounds}{\partial K} = \pi'\left(K\right) - \varphi u'\left(\pi\left(\beta K\right)\right)\pi'\left(\beta K\right)\beta\left(T - t\right) \tag{83}$$

$$-\dot{\rho} = \frac{\partial \pounds}{\partial U} = \varphi \geq 0 \tag{84}$$

The transversality conditions are as follows.

Since the constraint (14) is not binding for $t = 0$, as shown in equation (78), we must have

$$\rho\left(0\right) = 0, \tag{85}$$

and since $K\left(T\right)$ is free, we must have

$$\psi\left(T\right) = 0. \tag{86}$$

Condition (83) implies

$$\lim_{t \to T} \dot{\psi} = -\pi'\left(K\left(T\right)\right) < 0, \tag{87}$$

and thus, it follows from (86) and (87) that $\psi\left(t\right) > 0$ over some interval $\left(T - \varepsilon, T\right)$ and, using (79) and $v'\left(1\right) = 0$, we have $k\left(T\right) = 0$.

The constraint $U\left(t\right) \geq R\left(t\right)$ is binding at $t = t_a$. Then it must be binding over some interval $\left[t_a, t_b\right]$ (the interval may be degenerate):

$$U\left(t\right) = R\left(t\right) = u\left(\pi\left(\beta K\right)\right)\left(T - t\right) \text{ over } \left[t_a, t_b\right].$$

There are two possibilities: either $t_b > t_a$ (i.e. the interval is not degenerate), or $t_b = t_a$ (i.e. the interval is degenerate).

Suppose $t_b > t_a$. From time $t = 0$ to time $t = t_a$, because of the complementary slackness condition (82), and using (78), we have $\varphi\left(t\right) = 0$ for all $t \in \left[0, t_a\right)$. This implies, via (85) and (84), that $\dot{\rho}\left(t\right) = 0$ and $\rho\left(t\right) = \rho\left(0\right) = 0$ for all $t \in \left[0, t_a\right)$. This in turn implies, via (80) and (81), that $\alpha\left(t\right) = 1$ and $w\left(t\right) = 0$ for all $t < t_a$.

After t_a, $\rho\left(t\right)$ becomes negative (see (84) and (85)), and from (80), we can see that this permits $\alpha\left(t\right)$ to eventually equal zero after t_a. There are two subcases: (1) either $U\left(t\right)$ and (hence) $R\left(t\right)$ is constant over $\left[t_a, t_b\right]$, or (2) $U\left(t\right)$ and (hence) $R\left(t\right)$ are decreasing over $\left[t_a, t_b\right]$ (recall that, $\dot{U}\left(t\right) = -u\left(w\left(\tau\right)\right) \leq 0$).

In subcase (1), $U\left(t\right)$ and (hence) $R\left(t\right)$ are constant over $\left[t_a, t_b\right]$. This means that $w\left(t\right) = 0$ over $\left[t_a, t_b\right]$.Then, condition (80) can be rewritten as (if $\lim_{w \to 0} u'\left(w\right) < +\infty$)

$$-1 - \rho\left(t\right)u'\left(0\right) + \alpha\left(t\right) = 0 \text{ for all } t \in \left[t_a, t_b\right]. \tag{88}$$

Therefore,

$$-\dot{\rho}\left(t\right)u'\left(0\right) + \dot{\alpha}\left(t\right) = 0 \text{ for all } t \in \left[t_a, t_b\right], \tag{89}$$

i.e., using (84),

$$\dot{\alpha}(t) = \dot{\rho}(t)\, u'(0) = -\varphi(t)\, u'(0) \leq 0.$$

Let

$$C \equiv R(t_a) = R(t_b) = R(t) = (T-t)\, u(\pi(\beta K(t))),$$

for all $t \in [t_a, t_b]$.

The constancy of R implies

$$u(\pi(\beta K(t))) = \frac{C}{T-t} \text{ for all } t \in [t_a, t_b].$$

Then, differentiating with respect to t,

$$u'(\pi(\beta K(t)))\, \pi'(\beta K(t))\, \beta k(t) = \frac{C}{(T-t)^2} > 0.$$

Thus k is positive over $[t_a, t_b]$. Differentiating once more with respect to t, we have

$$(u'')(\pi'\beta k)^2 + (u')\left[(\pi'')(\beta k)^2 + \pi'\beta\dot{k}\right] = \frac{2C}{(T-t)^3},$$

or,

$$(u')\left[(\pi'')(\beta k)^2 + \pi'\beta\dot{k}\right] = \frac{2C}{(T-t)^3} - (u'')(\pi'\beta k)^2,$$

or,

$$\pi'\beta\dot{k}(t) = \frac{1}{u'}\left[\frac{2C}{(T-t)^3} - (u'')(\pi'\beta k)^2\right] - (\pi'')(\beta k)^2 > 0.$$

It follows that our result that $\dot{k} > 0$ over some interval (t', t'') continues to hold in this case.

Remark: If $\lim_{w\to 0} u'(w) = +\infty$, (the condition is satisfied for the CRRA utility function, $u(w) = w^{1-\sigma}/(1-\sigma)$, where $\sigma \in (0,1)$), subcase (1) is impossible. Indeed, condition (80), for $w \to 0$ becomes:

$$-1 - \rho(t) \lim_{w\to 0} u'(w) + \alpha(t) = 0 \text{ for all } t \in [t_a, t_b],$$

and then

$$\alpha(t) = -\infty \text{ for all } t \in [t_a, t_b],$$

which is impossible.

In subcase (2), $U(t)$ and (hence) $R(t)$ are decreasing over $[t_a, t_b]$. Then, over $[t_a, t_b]$, we have

$$-u(w(t)) = \dot{U}(t) < 0,$$

which implies $w(t) > 0$ over $[t_a, t_b]$ (because $u(0) = 0$ and $u' > 0$). Hence, using (81), we have $\alpha(t) = 0$ over $[t_a, t_b]$ and then (80) gives

$$-1 - \rho(t)\, u'(w(t)) = 0,$$

over $[t_a, t_b]$.

Differentiating this condition with respect to t, we find

$$-\dot{\rho}(t)\, u'(w(t)) - \rho(t)\, u''(w(t))\, \dot{w}(t) = 0.$$

Thus, since $-\dot{\rho}(t) = \varphi(t)$ and $\rho(t) = \frac{-1}{u'(w(t))}$ we have

$$\dot{w}(t) = \frac{-\dot{\rho}(t)\, u'(w(t))}{\rho(t)\, u''(w(t))} = \frac{\varphi(t)\,[u'(w(t))]^2}{-u''(w(t))} > 0, \tag{90}$$

i.e., wage is increasing over $[t_a, t_b]$.

Using (83) and differentiating (79) with respect to time, we have

$$-\dot{\psi} = \pi'(K) - \varphi u'(\pi(\beta K))\, \pi'(\beta K)\, \beta(T-t) = -v''(1-k)\dot{k},$$

Thus,

$$\varphi = \frac{\pi'(K) + \dot{k}v''(1-k)}{u'(\pi(\beta K))\, \pi'(\beta K)\, \beta(T-t)}. \tag{91}$$

Substituting into (90), we have

$$\frac{-u''(w(t))\, \dot{w}(t)}{[u'(w(t))]^2} = \frac{\pi'(K) + \dot{k}v''(1-k)}{u'(\pi(\beta K))\, \pi'(\beta K)\, \beta(T-t)}. \tag{92}$$

Since $U(t) = R(t)$ over $[t_a, t_b]$, we have

$$\int_t^T u(w(s))\, ds = u(\pi(\beta K))\,(T-t) \;\text{ over }\; [t_a, t_b].$$

Differentiating this condition with respect to time, we find

$$-u(w(t))\, \dot{w}(t) = u'(\pi(\beta K))\, \pi'(\beta K)\, \beta k(T-t) - u(\pi(\beta K)) \;\text{ over }\; [t_a, t_b]. \tag{93}$$

Let us consider the following example: $u(w) = (w)^{1-\sigma}/(1-\sigma)$ where $\sigma \in (0,1)$, $\pi(X) = AX$ and $v(x) = x - \frac{1}{2}x^2$. Conditions (92) and (93) become

$$\begin{aligned} (w)^{\sigma-1}\, \dot{w}(t) &= \frac{A-\dot{k}}{\sigma(A\beta)^{1-\sigma}(K)^{-\sigma}(T-t)}, \\ (w)^{1-\sigma}\, \dot{w}(t) &= (A\beta)^{1-\sigma}(K)^{-\sigma}\left[K - (1-\sigma)(T-t)\, k\right]. \end{aligned} \tag{94}$$

Multiplying the two equalities and dividing the second equality over the first one, we have

$$(\dot{w}(t))^2 = \frac{\left(A-\dot{k}\right) B}{\sigma(T-t)}, \tag{95}$$

$$(w(t))^{2(1-\sigma)} = A\beta\frac{\sigma(K)^{-2\sigma}(T-t)\, B}{A-\dot{k}}, \tag{96}$$

where $B = K - (1-\sigma)(T-t)k$.

Equations (95) and (96) can be written as $\dot{w} = g\left(t, K, k, \dot{k}\right)$ and $w = f\left(t, K, k, \dot{k}\right)$. Differentiating the second equation with respect to time, we have $\dot{w} = f_t\left(t, K, k, \dot{k}\right) + f_K\left(t, K, k, \dot{k}\right)k + f_k\left(t, K, k, \dot{k}\right)\dot{k} + f_{\dot{k}}\left(t, K, k, \dot{k}\right)\ddot{k}$, where subscripts denote partial derivatives. Hence, $f_t\left(t, K, k, \dot{k}\right) + f_K\left(t, K, k, \dot{k}\right)k + f_k\left(t, K, k, \dot{k}\right)\dot{k} + f_{\dot{k}}\left(t, K, k, \dot{k}\right)\ddot{k} = g\left(t, K, k, \dot{k}\right)$, a non-autonomous third order differential equation with respect to K. There are two boundary conditions: $K(0) = 0$ and $\dot{K}(T) = k(T) = 0$. This is a typical two-point boundary value problem, with a third order (non-linear) scalar differential equation. It is known that this kind of mathematical problem may have no, one or multiple solutions (see Lin *et al.* 2007 or Tian and Ge 2010, for instance). Our specific problem may have multiple solutions because we have to solve a third order differential equation with only two boundary conditions.

Appendix B: Material for the examples

In this appendix, we derive the optimal solution of the second best problem, when the initial stock of knowledge is $K(0) = K_0 \geq 0$, the discount rate is null, $r = 0$, and we specify $\pi(X) = AX$ and $v(l) = l - \frac{1}{2}l^2$. We then have $\pi'(X) = A$ and $v'(1-k) = k$.

Looking at a solution such that k is interior, the necessary conditions include

$$-k + \psi = 0, \tag{97}$$

$$-\rho + \alpha = 0, \tag{98}$$

$$\alpha \geq 0,\ w \geq 0,\ \alpha w = 0 \tag{99}$$

$$\varphi \geq 0,\ W - R \geq 0,\ \varphi(W - R) = 0 \tag{100}$$

$$-\dot{\psi} = A - \varphi A \beta (T - t) \tag{101}$$

$$-\dot{\rho} = \varphi \tag{102}$$

The following transversality conditions are also necessary. Since K_T and W_0 are free,

$$\psi(T) = 0, \tag{103}$$

$$\rho(0) = 1. \tag{104}$$

We have also $K(0) = K_0 \geq 0$ and $W(T) = 0$.

We look at the optimal solution such that there is only one phase $[t_a, t_b]$ where k is increasing. From the proof of Proposition 2, condition (23), we know that, for all $t \in [0, t_a] \cup [t_b, T]$,

$$\dot{k}(t) = \frac{-\pi'(K(t))}{-v''(1 - k(t))} = -A \tag{105}$$

Integrating over $[t, t_a]$ with $t \in [0, t_a]$, we find

$$k(t) = k(t_a) + A(t_a - t), \tag{106}$$

for all $t \in [0, t_a]$.

Integrating (105) over $[t_b, t]$ with $t \in [t_b, T]$, we find

$$k(t) = k(t_b) - A(t - t_b). \tag{107}$$

We also know from the proof of Proposition 2, condition (22),

$$\begin{aligned} \dot{k} &= \frac{2\pi'(\beta K) k + \beta (T - t)(-\pi''(\beta K)) k^2}{(T - t)\pi'(\beta K)} \\ &= \frac{2k}{T - t}, \end{aligned} \tag{108}$$

for all $t \in [t_a, t_b]$.

Integrating over $[t_a, t]$ with $t \in [t_a, t_b]$ and rearranging, we find

$$k(t) = k(t_a)\left(\frac{T - t_a}{T - t}\right)^2. \tag{109}$$

Integrating (106) over $[t, t_a]$ with $t \in [0, t_a]$, we have

$$K(t) = K(t_a) - \left[\frac{A}{2}(t_a - t) + k(t_a)\right](t_a - t), \tag{110}$$

and then

$$K_0 = K(t_a) - \left[\frac{A}{2}t_a + k(t_a)\right]t_a, \tag{111}$$

Integrating (109) over $[t_a, t]$ with $t \in [t_a, t_b]$, we find

$$K(t) = K(t_a) + k(t_a)(T - t_a)^2\left(\frac{1}{T - t} - \frac{1}{T - t_a}\right). \tag{112}$$

Integrating (107) over $[t_b, t]$ with $t \in [t_b, T]$, we find

$$K(t) = K(t_b) + k(t_b)(t - t_b) - \frac{1}{2}A(t - t_b)^2. \tag{113}$$

It remains to solve for $(t_a, t_b, k(t_a), k(t_b), K(t_a), K(t_b))$.

By continuity of k at $t = t_b$ and using (109), we have:

$$k(t_b) = k(t_a)\left(\frac{T - t_a}{T - t_b}\right)^2, \tag{114}$$

and by continuity of K at $t = t_b$ and using (112), we find:

$$K(t_b) = K(t_a) + k(t_a)(T - t_a)^2\left(\frac{1}{T - t_b} - \frac{1}{T - t_a}\right). \tag{115}$$

From the proof of Proposition 2, we know that $R(t)$ is constant over $[t_a, t_b]$ and then,

$$K(t_a) = \frac{T - t_b}{T - t_a} K(t_b). \tag{116}$$

Using (103) and (97), we have $k(T) = 0$. Hence, using (107) we obtain:

$$k(t_b) = A(T - t_b). \tag{117}$$

Using (97), (101) and (102), we have

$$\dot{\rho} = -\frac{\dot{k} + A}{A\beta(T - t)}, \tag{118}$$

and integrating over $[0, t]$ with $t \in [0, t_a]$ and using $\rho(0) = 1$ and (105), we find

$$\rho(t) = 1, \tag{119}$$

for all $t \in [0, t_a]$.

Using (108) and (109) in (118) and integrating over $[t_a, t]$ with $t \in [t_a, t_b]$, and using (119), we find

$$\begin{aligned} \rho(t) &= \rho(t_a) - \frac{1}{A\beta} \int_{t_a}^{t} \frac{\dot{k}(\tau) + A}{T - \tau} d\tau \\ &= 1 - \frac{1}{\beta} \int_{t_a}^{t} \left(2\frac{k(t_a)}{A} \frac{(T - t_a)^2}{(T - \tau)^4} + \frac{1}{T - \tau} \right) d\tau, \end{aligned}$$

or,

$$\rho(t) = 1 - \frac{2}{3\beta A} k(t_a)(T - t_a)^2 \left(\frac{1}{(T - t)^3} - \frac{1}{(T - t_a)^3} \right) + \frac{1}{\beta} \ln\left(\frac{T - t}{T - t_a} \right) \tag{120}$$

Using (105) and integrating (118) over $[t_b, t]$ with $t \in [t_b, T]$, we find

$$\rho(t) = \rho(T), \tag{121}$$

for all $t \in [t_b, T]$.

Let us show that $\rho(T) = 0$. Suppose the reverse, i.e. $\rho(T) > 0$. Using (121), we have $\rho(t) > 0$ for all $t \in [t_b, T]$. Using (98) and (99), this implies that $w(t) = 0$ for all $t \in [t_b, T]$. However, we know from the proof of Proposition 2 that $w(t) = 0$ for all $t \in [0, t_b]$. Hence $w(t) = 0$ for all t, but we know this is impossible because we have assumed that $k(t) > 0$ for $t < T$ (see again the proof of Proposition 2). This proves that $\rho(T) = 0$.

By continuity of ρ at $t = t_b$ and using (120) and (121), we then have

$$\rho(T) = 0 = \rho(t_b) = 1 - \frac{2}{3\beta A} k(t_a)(T - t_a)^2 \left(\frac{1}{(T - t_b)^3} - \frac{1}{(T - t_a)^3} \right) + \frac{1}{\beta} \ln\left(\frac{T - t_b}{T - t_a} \right),$$

or,

$$A\beta = -A \ln\left(\frac{T - t_b}{T - t_a} \right) + \frac{2}{3} k(t_a)(T - t_a)^2 \left(\frac{1}{(T - t_b)^3} - \frac{1}{(T - t_a)^3} \right). \tag{122}$$

By solving (114), (115), (116), (117), (111) and (122), we can find t_a, t_b, $k(t_a)$, $k(t_b)$, $K(t_a)$, and $K(t_b)$.

Rearranging these conditions, and let $\Delta \equiv \frac{T-t_b}{T-t_a}$, we find $k(t_b) = A(T-t_b)$; $k(t_a) = A\frac{(T-t_b)^3}{(T-t_a)^2}$; $K(t_a) = A\frac{(T-t_b)^3}{T-t_a}$; and $K(t_b) = A(T-t_b)^2$, and

$$A\Delta^3 = \frac{2K_0 + A(t_a)^2}{2(T-2t_a)(T-t_a)}, \tag{123}$$

and,

$$\beta = -\ln(\Delta) + \frac{2}{3}\left(1-\Delta^3\right). \tag{124}$$

It is easy to show that (124) has a unique solution in $(0,1)$, because $\beta > 0$ and $\Delta \mapsto -A\ln(\Delta) + \frac{2}{3}\left(1-\Delta^3\right)$ is decreasing and it goes to $+\infty$ when $\Delta \to 0$ and it is 0 for $\Delta = 1$. Notice that we need to have $\frac{2K_0+A(t_a)^2}{2A(T-2t_a)(T-t_a)} \leq 1$, or $K_0 \leq A(T-2t_a)(T-t_a) - \frac{1}{2}A(t_a)^2$.

Finally, we can write the optimal path of the share of time devoted to knowledge accumulation as follows:

$$k(t) = \begin{cases} A\frac{(T-t_b)^3}{(T-t_a)^2} + A(t_a - t) \text{ for } t \in [0, t_a], \\ A(T-t_b)^3 (T-t)^{-2} \text{ for } t \in [t_a, t_b], \\ A(T-t) \text{ for } t \in [t_b, T], \end{cases}$$

and the optimal cumulated knowledge is

$$K(t) = \begin{cases} K_0 + \left(A\frac{(T-t_b)^3}{(T-t_a)^2} + At_a - A\frac{t}{2}\right) t \text{ if } t \in [0, t_a] \\ A\frac{(T-t_b)^3}{T-t} \text{ if } t \in [t_a, t_b] \\ A(T-t_b)^2 + A(t-t_b)\left(T - \frac{1}{2}(t+t_b)\right) \text{ if } t \in [t_b, T] \end{cases}$$

■

Appendix C: Sufficiency

In this Appendix, we show that our solution satisfies the sufficiency condition and therefore dominates all other feasible solutions. The proof applies to the case without discounting as well as the case with a positive discount rate r and $K_0 \geq 0$.

Sufficiency Theorem: *Let* $(K^*, Z^*, k^*, z^*, K_T^*, Z_0^*)$ *be a candidate optimal solution, with the associated time path of shadow prices* $(\psi^*, \rho^*, \lambda^*, \mu^*, \alpha^*, \varphi^*)$. *Assume that all the necessary conditions (including the transversality conditions) are satisfied. Consider any alternative feasible plan* $(K^\#, Z^\#, k^\#, z^\#, K_T^\#, Z_0^\#)$. *Let*

$$\begin{aligned} L(K, Z, k, z, \psi^*, \rho^*, \lambda^*, \mu^*, \alpha^*, \varphi^*, t) \quad \equiv \quad & e^{-rt}\pi(K) + e^{-rt}v(1-k) + \psi^* k + \rho^* z + \\ & \lambda^* k + \mu^*[1-k] + \alpha^* z + \varphi^*(Z - \pi(\beta K)\theta) \end{aligned}$$

Let V_f^* *and* $V_f^\#$ *be the payoffs obtained by carrying the plans* $(K^*, Z^*, k^*, z^*, K_T^*, Z_0^*)$ *and* $(K^\#, Z^\#, k^\#, z^\#, K_T^\#, Z_0^\#)$ *respectively. Assume that* L *is concave in* (K, Z, k, z). *Then* $V_f^* \geq V_f^\#$.

Proof: Our proof is similar to that of Takayama (1986).

For simplicity, we use the following notations

$$L^* = L(K^*, Z^*, k^*, z^*, \psi^*, \rho^*, \lambda^*, \mu^*, \alpha^*, \varphi^*, t)$$

and

$$L^\# = L(K^\#, Z^\#, k^\#, z^\#, \psi^*, \rho^*, \lambda^*, \mu^*, \alpha^*, \varphi^*, t)$$

where the asterisk over the multipliers indicates that we use the same path $(\psi^*, \rho^*, \lambda^*, \mu^*, \alpha^*, \varphi^*)$ for both L^* and $L^\#$.

Since $\lambda^* k^* = 0$, $\mu^* [1 - k^*] = 0, \alpha^* z^* = 0$ and $\varphi^* [Z^* - \pi(\beta K^*)\theta] = 0$,

$$V_f^* = -Z_0^* + \int_0^T \left[L^* - \psi^* \dot{k}^* - \rho^* \dot{z}^*\right] dt$$

Now,

$$V_f^\# = -Z_0^\# + \int_0^T \left[L^\# - \psi^* \dot{k}^\# - \rho^* \dot{z}^\# - \lambda^* k^\# - \mu^* \left[1 - k^\#\right] - \alpha^* z^\# - \varphi^* \left(Z^\# - \pi(\beta K^\#)\theta\right)\right] dt$$

And, since $\lambda^* \geq 0, \mu^* \geq 0, \alpha^* = 0$ and $\varphi^* \geq 0$, and since feasibility requires that $k^\# \geq 0$, $1 - k^\# \geq 0, z^\# \geq 0$ and $Z^\# - \pi(\beta K^\#)\theta \geq 0$

$$V_f^\# \leq -Z_o^\# + \int_0^T \left[L^\# - \psi^* \dot{k}^\# - \rho^* \dot{z}^\#\right] dt$$

Then

$$\begin{aligned} V_f^* - V_f^\# \quad &\geq \quad -\left(Z_0^* - Z_0^\#\right) - \int_0^T \left[\psi^* \dot{k}^* - \psi^* \dot{k}^\#\right] dt - \int_0^T \left[\rho^* \dot{z}^* - \rho^* \dot{z}^\#\right] dt \\ &\quad + \int_0^T \left[L^* - L^\#\right] dt \end{aligned}$$

Now, under the assumption that L is concave in (K^*, Z^*, k^*, z^*),

$$L^* - L^\# \geq \left(k^* - k^\#\right) \frac{\partial L^*}{\partial k^*} + (K^* - K^\#) \frac{\partial L^*}{\partial K^*} +$$

$$(Z^* - Z^\#) \frac{\partial L^*}{\partial Z^*} + \left(z^* - z^\#\right) \frac{\partial L}{\partial z^*}$$

Now, from the necessary conditions $\frac{\partial L^*}{\partial k^*} = \frac{\partial L^*}{\partial z^*} = 0$, $\frac{\partial L^*}{\partial K^*} = -\dot{\psi}^*$and $\frac{\partial L^*}{\partial Z^*} = -\dot{\rho}^*$, we have

$$L^* - L^\# \geq -\dot{\psi}^* (K^* - K^\#) - \dot{\rho}^* (Z^* - Z^\#)$$

Therefore

$$V_f^* - V_f^\# \geq -\left(Z_0^* - Z_0^\#\right) - \int_0^T \left[\dot{\psi}^* (K^* - K^\#) + \psi^* \dot{K}^* - \psi^* \dot{K}^\#\right] dt +$$

$$- \int_0^T \left[\dot{\rho}^* (Z^* - Z^\#)\right] dt + \int_0^T \rho^* \dot{Z}^* - \rho^* \dot{Z}^\# dt$$

$$\geq \quad -\left(Z_0^* - Z_0^{\#}\right) - [\psi^*(T)K_T^* - \psi^*(0)K_0^*] + \left[\psi^*(T)K_T^{\#} - \psi^*(0)K_0^{\#}\right]$$
$$- [\rho^*(T)Z^*(T) - \rho^*(0)Z_0^*] + \left[\rho^*(T)Z^{\#}(T) - \rho^*(0)Z_0^{\#}\right]$$

Using the transversality conditions $\psi^*(T) = 0 = \rho^*(T)$ and the fixed initial condition, $K_0^* = K_0^{\#} = K_0$, the above inequality becomes $V_f^* - V_f^{\#} \geq [\rho^*(0) - 1]\left(Z_0^* - Z_0^{\#}\right)$. But we have shown that $[\rho^*(0) - 1] = 0$. It follows that $V_f^* - V_f^{\#} \geq 0$. ■

Figures

First best

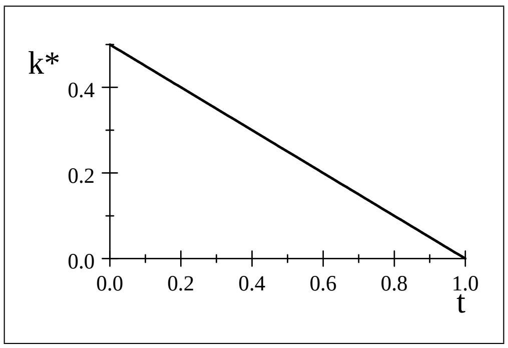

Figure 1.a: First best allocation of working time

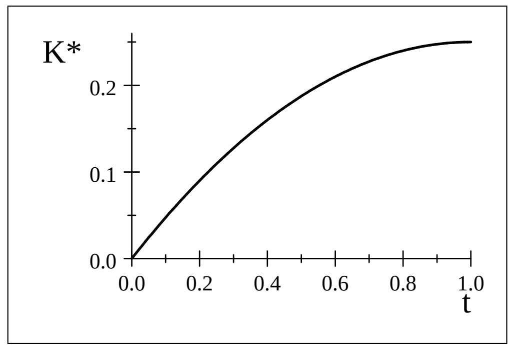

Figure 1.b: First best cumulated knowledge

Second best

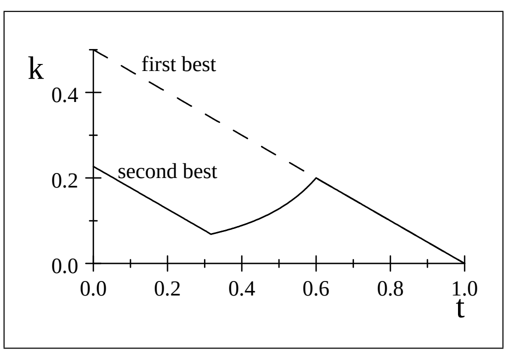

Figure 2.a: Knowledge task share

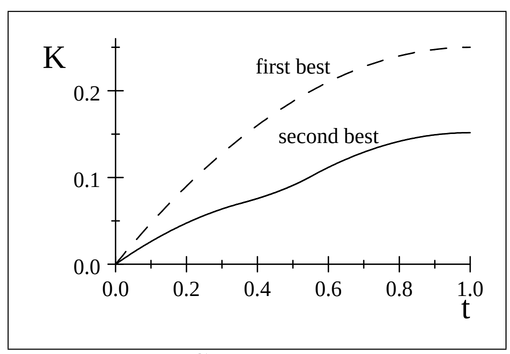

Figure 2.b: Cumulated knowledge

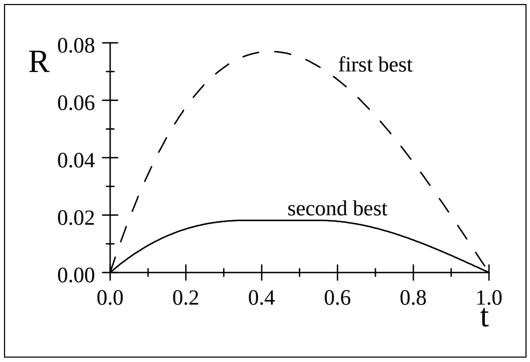

Figure 2.c: Knowledge worker's outside option

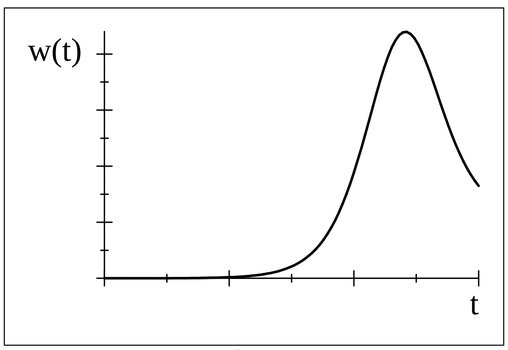

Figure 3.a: Optimal wages

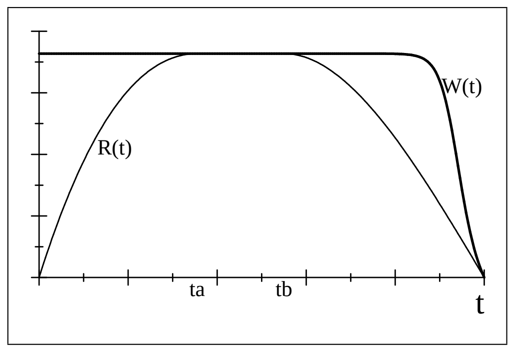

Figure 3.b: Outside option and remaining earnings

Life-cycle wage profiles

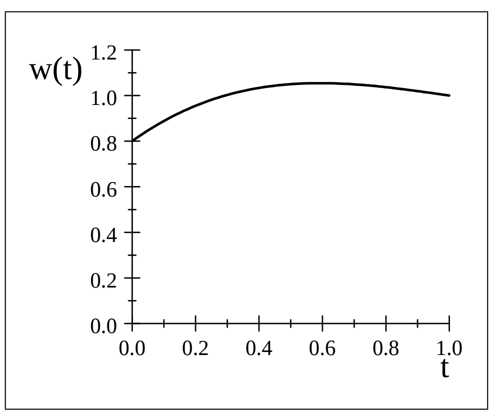

Figure 3.c: Life-cycle wage profiles

Severity of the enforcement problem (β) or knowledge specificity $(1 - \beta)$

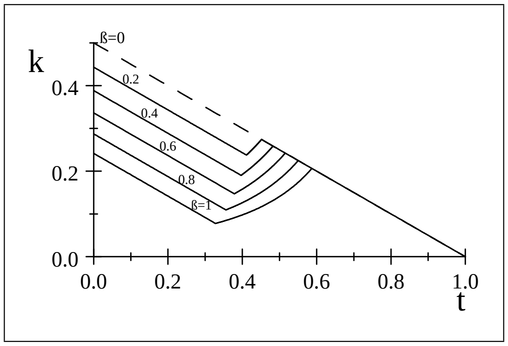

Figure 4.a: Knowledge task share of working time and β

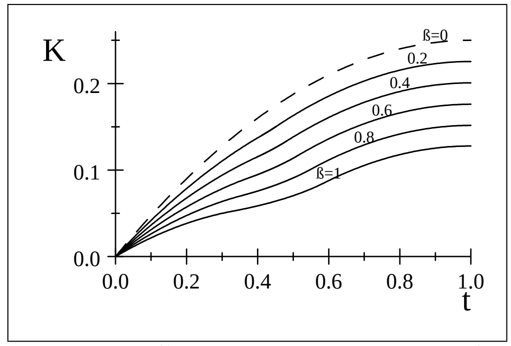

Figure 4.b: Cumulated knowledge and β

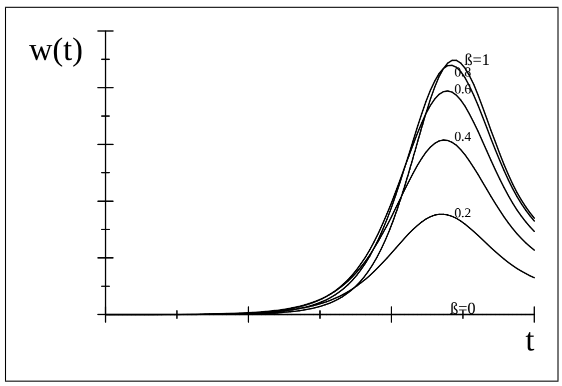

Figure 4.c: Optimal wages and β.

Role of initial knowledge level $(K_0 \geq 0)$

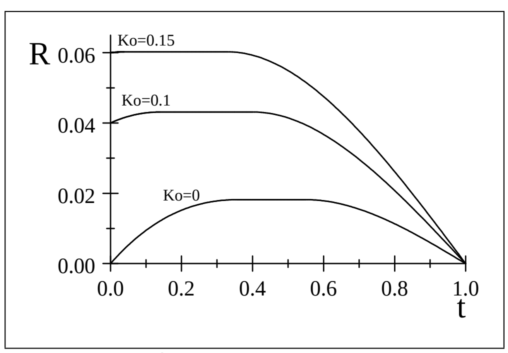

Figure 5.a: Outside option and $K_0 \geq 0$.

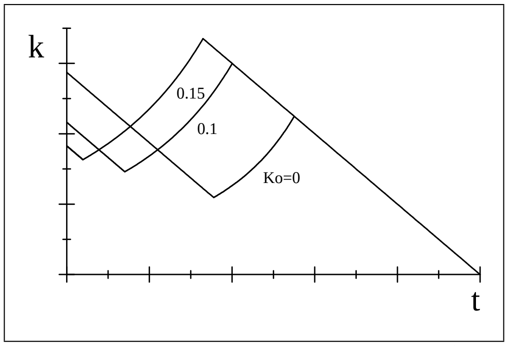

Figure 5.b: Knowledge task share of working time and $K_0 \geq 0$.

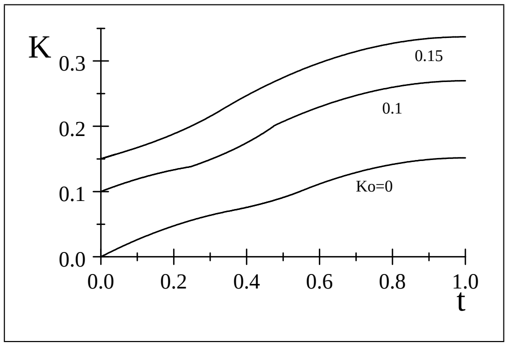

Figure 5.c: Cumulated knowledge and $K_0 \geq 0$.

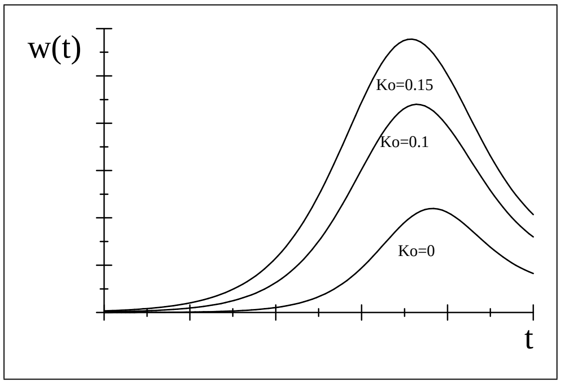

Figure 5.d: Optimal wages and $K_0 \geq 0$.

Role of discount ($r \geq 0$)

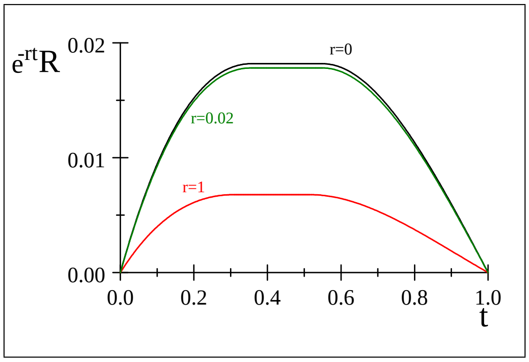

Figure 6.a: Knowledge worker's outside option and $r \geq 0$.

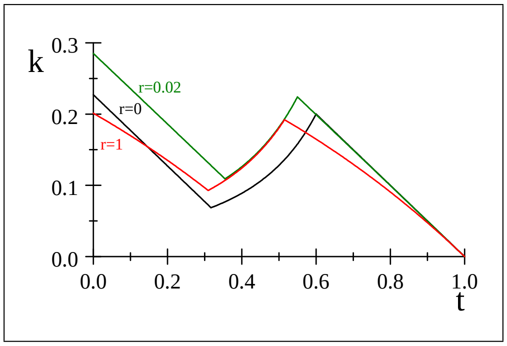

Figure 6.b: Knowledge task share and $r \geq 0$.

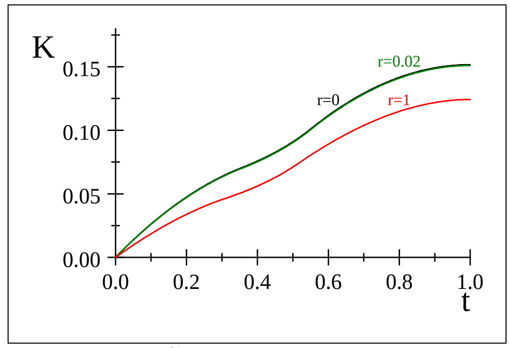

Figure 6.c: Cumulated knowledge and $r \geq 0$.

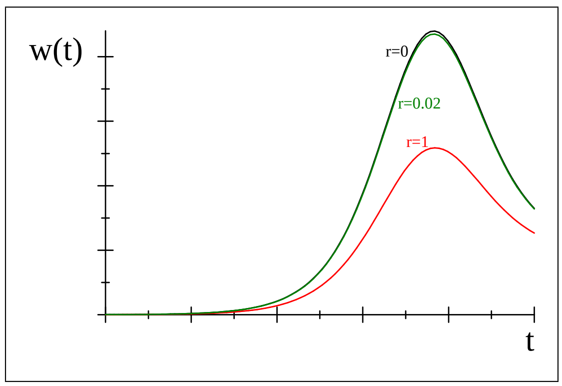

Figure 6.d: Optimal wages and $r \geq 0$.

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