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The Mixed Bentham-Rawls Intertemporal Choice Criterion and Rawls' Just Savings Principle

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Résumé/Abstract

This paper provides general theorems about the control that maximizes the mixed Bentham-Rawls (MBR) criterion for intergenerational justice, which was introduced in Alvarez-Cuadrado and Long (2009). We establish sufficient concavity conditions for a candidate trajectory to be optimal and unique. We show that the state variable is monotonic under rather weak conditions. We also prove that inequality among generations, captured by the gap between the poorest and the richest generations, is lower when optimization is performed under the MBR criterion rather than under the discounted utilitarian criterion. A quadratic example is also used to perform comparative static exercices: it turns out, in particular, that the larger the weight attributed to the maximin part of the MBR criterion, the better-off the less fortunate generations. All those properties are discussed and compared with those of the discounted utilitarian (DU, Koopmans 1960) and the rank-discounted utilitarian (RDU, Zuber and Asheim, 2012) criterions. We contend they are in line with some aspects of the rawlsian just savings principle.

Mot clés/Keywords: intergenerational equity, just savings principle.

Codes JEL/JEL Codes: D63, H43, O21, Q20

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1 Introduction

This article gets back to Rawls' *just savings principle* (Rawls, 1971, 1999) and to its link with the recent mixed Bentham-Rawls (MBR) intertemporal choice criterion (Long 2006, Alvarez-Cuadrado & Long, 2009, Tol, 2013). In a standard dynamic model, we elucidate general properties of the control path that is deemed optimal according to the MBR criterion; those properties turn out to be, to some extent, consistent with some intuitions about intergenerational justice expressed *via* Rawls' just savings principle.

The general challenge is to think about what we owe to future generations[1]. One possible answer, the *just savings principle* suggested by Rawls and various scholars working in this field, derives from the *social contract* approach initiated by Grotius, Hobbes, Locke, Rousseau and Kant. It can be described as the saving rule that an impartial "observer" would deem fair. It advocates a two-phase logic (Gosseries, 2001, Wall, 2003). During a first phase, each generation must save in order to transfer to the next generation more than what it has inherited from the previous generations; the purpose of the accumulation phase is to build up economic conditions so that at least basic freedoms are in place and minimal stability to just institutions can be ensured. Then follows a cruise phase where the principle of equality that prevails recalls the *egalitarian-maximin* logic, but it is subordinate to the need for an initial take-off from the condition of under-development[2]. Qualitatively, the picture is clear; but when it comes to more operational details, the just savings principle has proven somewhat elusive[3]. Its implications are far too vague and require further precisions. Unfortunately, Rawls did not suggest a precise criterion that would embody his just savings clause.

The MBR criterion has been suggested as a possibility, where the impartial observer would be a 'Dynasty with Concern for the Least Advantaged' (Long, 2006). And indeed, its application in particular contexts abide by a two-phase logic when the initial conditions are too low, that is to say below the modified golden-rule (Long, 2006, Alvarez-Cuadrado & Long, 2009). But, beyond those contexts, could this dynamic pattern be a robust property of the optimal path according the MBR criterion? And, since this property also characterizes the optimal control for the widely used discounted utilitarian criterion, what is the further argument in favour of the MBR criterion as a superior embodiment of the just savings principle?

[1]Since the famous UN Brundtland Report (1987), "Our Common Future", and its echo in the political sphere, it is a question that is experiencing a heightened interest; but philosophers - and also economists - have thought about it for a long time.

[2]"Eventually, once just institutions are firmly established and all the basic liberties effectively realized, the net accumulation asked for falls to zero. At this point a society meets its duty of justice by maintaining just institutions and preserving their material base." (Rawls, 1999).

[3]Several intertemporal social choice criteria used by economists have the ability to prescribe this two-phase logic, though they also imply dramatically different saving rates. This is the case for the "distance-to-bliss" criterion (Ramsey, 1928), the usual discounted utilitarian criterion (Koopmans, 1960) and the rank-discounted utilitarian criterion (Zuber and Asheim, 2012).

This article is organized as follows. The next section gives further details about Rawls' just savings principle. Section 3 presents a simple - but general - dynamic economy that encompasses various applications, and it also explains the MBR criterion. Section 3 establishes and discusses general dynamic properties of the MBR exploitation path; and it also shows that inequalities, captured by the difference of utility between the richest and the poorest generations, is generically lower under the MBR criterion than under the discounted utilitarian criterion. Section 4 presents a fully worked-out example, in which we prove analytically three additional results. First, a greater initial stock level implies a greater consumption for the poorest generations. Second, a greater weight on the rawlsian part of the MBR criterion makes the poorest generations better-off. Third, the larger the weight on the rawlsian part of the MBR criterion, the longer the initial egalitarian phase of the MBR optimal program. Section 5 concludes.

2 The just savings principle

Chapter V of *A Theory of Justice* (Rawls, 1999, revised edition) contains three sections of particular interest when it comes to justice between generations. Section 44 is entirely devoted to this question. Section 45 discusses the link between the just savings principle and the notion of time preference. Finally, Section 46 incorporates the concern for just savings in the overall project of characterizing just institutions. Below are some highlights of this thinking.

The general purpose of the just savings principle is to ensure that each generation receives its due from its predecessors and does its fair share for those to come, with the understanding that the savings rate of each is to serve the whole span of accumulation in order to establish just institutions and realize basic liberties. Differently put, the just savings principle says that when the society has not accumulated enough wealth to ensure the foundation of a solid infrastructure for justice and liberty, then earlier generations should save so that society has a sufficient material base to develop such an infrastructure[4]. This principle does not say precisely how much saving is required of each of the earlier generations. Rawls contends that this question "seems to admit of no definite answers. It does not follows, however, that certain significant ethical constraints cannot be formulated" (p. 253). Rawls just hopes that certain extremes will be excluded. One such extreme is typically given by the optimal saving path derived from the undiscounted utilitarian criterion that, by construction, generally demands exorbitant sacrifices to the first generations. Indeed, when investment has a positive return, a sacrifice by the current generation often appears worthwhile

[4]From Rawls' contractuarian perspective, "the just savings principle can be regarded as an understanding between generations to carry their fair share of the burden of realizing and preserving a just society." (p. 257.) And he goes on to say "The [just] savings principle represents an interpretation, arrived at in the original position, of the previously accepted natural duty to uphold and to further just institutions. In this case the ethical problem is that of agreeing on a path over time which treats all generations justly during the whole course of a society's history." (pp. 288-289).

when it is pitted against the infinite sum of undiscounted advantages it will produce on subsequent generations[5]. Interestingly, Rawls also expresses an objection against the use of his own *difference principle.* Directly transposed into the accumulation context, it would ask the wealth of the better-off generations to be scaled down and transferred to the poorest until eventually everyone ends up in the same situation. But an intertemporal context imposes specific constraints to such operations, for there is no way "for later generations to help the situation of the least fortunate earlier generation (p. 254)." The impossibility to organize transfers from descendants to ascendants implies some adjustment to the difference principle, except in special circumstances, presumably when the transfers required to abide by this principle naturally flow along the timeline: "what is just or unjust is how institutions deal with the natural limitations and the way they are set up to take advantages of historical possibilities (p. 254)".

In Section 45, Rawls clearly rejects the notion of time preference as a basic principle, even though he admits that weighting less heavily future generations may help improve otherwise defecting criteria. This practice may be used when later generations are richer, but he consider such adjustements as "an indication that we have started from an incorrect conception", for "time preference has no intrinsic ethical appeal" (p. 262).

In Section 46, Rawls is ready to deliver the final statements of justice for institutions, with due qualifications for the issue of savings. He repeats his two main principles (the equal liberty principle and the difference principle) and enriches the priority rules for potential dissonances among them. While the first principle is lexicographically prior to the second for matters of justice *within* generations, the savings principle limits the scope of the difference principle for matters of justice *among* generations.

3 A dynamic framework and the MBR criterion

Time is continuous and the horizon is infinite. The economy has infinitely many successive generations. Each generation is made of one representative individual who lives for just one instant. Let $c(t)$ be the control variable, or consumption flow, that affects generation t, and the stock variable is $x(t)$. This stock evolves according to the differential equation:

$$\begin{cases} \frac{dx(t)}{dt} \equiv \dot{x}(t) = f(x(t), c(t)), \\ x(0) = x_0 \text{ given.} \end{cases} \tag{1}$$

An *admissible path* $\{c(.), x(.)\}$ is a solution to (1) such that $x(t) \geq 0$ and $c(t) \geq 0, \forall t$.

When consuming $c(t)$, generation t enjoys a standard of living, $U(t) \equiv U(c(t))$, where $U(.)$ is an increasing function[6]. To any path $c(.)$ let $\underline{c} =$

[5] Under plausible specifications of the economy it may yield optimal savings amounting to more than 60 percent of gross national product for the first generations.

[6] Well-being experienced by generation t is only but one interpretation that could be given to function $U(.)$.

$\inf_t \{c(.)\}$ stand for the lowest consumption level, and let $\underline{U} = U(\underline{c})$ be the corresponding standard of living.

This framework can accommodate two standard interpretations: *i*) the Ramsey-Solow optimal growth model, that is obtained as a particular case when x is the capital stock and the dynamics are $\dot{x} = f(x, c) = F(x) - c - \delta x$, where $F(.)$ is a production function such that $F(x) \geq 0, F'(x) \geq 0, F''(x) < 0$, and $\delta > 0$ is the rate of depreciation, *ii*) the basic renewable resource model, when x is a natural resource that evolves according to the equation $\dot{x} = f(x, c) = G(x) - c$, where $G(.)$ is a concave function that reaches a maximum at some x^M, called the *maximum sustainable yield.*

To any admissible path, the *MBR criterion* associates the following value:

$$W^{mbr}(c(.)) \equiv \theta \underline{U} + (1-\theta) \int_{t=0}^{\infty} e^{-rt}\, U(c(t))\, dt \ , \quad 0 < \theta < 1 \ . \tag{2}$$

W^{mbr} is a weighted average of the *maximin criterion* and the usual *discounted utilitarian criterion.* It can be seen as a procedural compromise between the concern for the worse-off (the larger θ, the stronger this concern) and the concern for all generations with a discount for the position on the temporal axis.

Some properties of the MBR criterion are exposed and discussed at length in Alvarez-Cuadrado & Long (2009). From a deontologic point of view, let us recall briefly that W^{mbr} meets the following requirements: *completeness, strong Pareto, non-dictatorship of the future* and *non-dictatorship of the present*[7]. These properties are important, but we do not dwell on them here; they are widely discussed elsewhere (Alvarez-Cuadrado & Long, 2009), and in any case it is a consequentialist point of view that interests us in this article. To be more precise, the object of interest in the present paper is not the expression (2) itself; rather we shall focus on the properties of the solution to the *MBR problem, i.e.* the trajectory $\{c(.), x(.)\}$ that maximizes (2) subject to (1).

4 Properties of optimal trajectories under the MBR criterion

The economic framework presented above features a minimal structure. Yet, even this basic structure already implies the following property on the endogenous path induced by the MBR criterion.

Theorem 1 (Monotonicity) *Let the pair $\left\{c^{mbr}(.), x^{mbr}(.)\right\}$ be a solution to the MBR problem. Assume that $c^{mbr}(.)$ is not constant and $x^{mbr}(.)$ is unique. Then $x^{mbr}(.)$ is monotonic for $t \in [0, +\infty[$.*

[7] An intertemporal social function is *complete* if it can rank any pairs of admissible paths. It satisfies *Strong Pareto* if it is increasing in any $U_t \equiv U(c(t))$. It displays *dictatorship of the present* when its ranking is not sensitive to the utility of generations located after some date T. It displays *dictatorship of the future* when its ranking is affected only by the utility of generations that are infinitely distant (see Chichilnisky, 1996).

Proof. Appendix A. ■

Note that Theorem 1 does not rest on demanding assumptions on the fundamentals of the economy given by functions $f(.,.)$ and $U(.)$. Regarding $f(.,.)$, neither differentiability, nor continuity, nor Lipschitzian assumptions are really necessary. Those kinds of assumptions help to guarantee the existence a solution to the differential equation (1), but they are sufficient and not necessary. We don't need either any concavity assumptions on $U(.)$ or any transversality conditions. Such conditions are helpful to guarantee that candidate paths are indeed optimal, but they are only sufficient, hence too strong. Theorem 1 sheds light more directly on the existence of a possible structure on endogenous variables whereby the optimal stock x cannot be cyclical, or increasing and then decreasing and *vice versa*. Actually, this theorem is a generalisation of Hartl's result (1987) about the monotonicity of the state trajectories in autonomous control problems[8].

Nevertheless, sufficient conditions can be identified to ensure the MBR problem is strictly concave and therefore its solution is unique. For the sake of completeness, they are given in Theorem 2 below.

Theorem 2 (Sufficient conditions for optimality) *Let $\left(c^{mbr}(.), x^{mbr}(.), \underline{c}^{mbr}\right)$ be a candidate optimal solution, with the associated time path of shadow prices $\left(\psi^{mbr}(.), \lambda^{mbr}(.)\right)$. Assume that the following transversality conditions are satisfied:*

$$\lim_{t\to\infty} \psi^{mbr}(t)x^{mbr}(t) = 0 \ , \tag{3}$$

and

$$\lim_{t\to\infty} \psi^{mbr}(t) \geq 0 \ . \tag{4}$$

Consider any alternative admissible path $\left(c^{\#}(.), x^{\#}(.), \underline{c}^{\#}\right)$. For any $(c, x, \underline{c})$, we define the following Lagrangian using the shadow prices of the candidate optimal path:

$$\begin{aligned} L(c, x, \underline{c}, \psi^{mbr}, \lambda^{mbr}, t) \quad \equiv \quad & e^{-rt}\left[r\theta U(\underline{c}) + (1-\theta)U(c)\right] \\ & +\psi^{mbr} f(x,c) + \lambda^{mbr}\left(c - \underline{c}\right) \ . \end{aligned}$$

Let V^{mbr} and $V^{\#}$ be the payoffs obtained by implementing the paths $\left(c^{mbr}(.), x^{mbr}(.), \underline{c}^{mbr}\right)$ and $\left(c^{\#}(.), x^{\#}(.), \underline{c}^{\#}\right)$ respectively, i.e.

$$V^{mbr} = \int_0^\infty e^{-rt}\left[r\theta U(\underline{c}^{mbr}) + (1-\theta)U(c^{mbr}(t))\right] \, dt \ ,$$

[8] Hartl (1987) deals with the discounted utilitarian criterion that, strictly speaking, is a particular case of expression (2) only when $\theta = 0$, a value that is ruled out in principle. However this value is forbidden simply to ensure that MBR escapes the dictatorship of the present. But nothing in the proof of Theorem 1 is compromised when $\theta = 0$. The proof just becomes simpler.

$$V^{\#} = \int_0^\infty e^{-rt} \left[r\theta U(\underline{c}^{\#}) + (1-\theta)U(c^{\#}(t)) \right] \; dt \; .$$

Assume that L is concave in $(c, x, \underline{c})$. Then $V^{mbr} \geq V^{\#}$. In the case where L is strictly concave in $(c, x, \underline{c})$, then the optimal solution is unique.

Proof. Appendix B. ■

Two comments about this theorem are worthwhile[9]. Firstly, at a formal level the candidate path, which is alluded to in the above theorem, is akin to the competitive outcome when instantaneous "total utility" is $r\theta U(\underline{c})+(1-\theta)U(c(t))$ and an additional price $\lambda(t)$ on consumption appears in cases where consumption otherwise would have fallen below $\underline{c}$. Secondly, one may wish to know about the concavity conditions to be imposed on the fundamentals U and f instead of those imposed on the Lagrangian L. Note that when - as usually assumed - $\partial f/\partial c < 0$ then first order conditions implies $\psi > 0$. Therefore the concavity of U and f implies the concavity of L.

Added to the general necessary conditions already given in Alvarez-Cuadrado & Long (2009) (see also Appendix B of the present paper), the above theorem about sufficient conditions completes the "user kit" of the MBR criterion.

Next, let us assume in this dynamic framework that x is a *productive asset,* in the following sense:

Assumption 1 *For any pair of points in time (t_a, t_b), where $t_a < t_b$, and any non-negative initial stock level a, let $c^*(.)$ be an admissible control path in the time interval $[t_a, t_b]$, i.e.*

$$\begin{cases} \dot{x}(t) = f(x(t), c^*(t)) \;, \forall t \in [t_a, t_b] \;, \\ x(t_a) = a, \;\; x(t) \geq 0, \forall t \in [t_a, t_b] \;, \end{cases}$$

and let b be the resulting stock size at time t_b,

$$b \equiv x(t_a) + \int_{t_a}^{t_b} f(x(t), c^*(t)) dt \; .$$

Then, for any $\varepsilon > 0$, there exists an admissible path $c_\varepsilon(.)$ in the time interval $[t_a, t_b]$ with the corresponding initial stock $x(t_a) = a + \varepsilon$ such that

$$c_\varepsilon(t) \geq c^*(t) \text{ for all } t \in [t_a, t_b] \;,$$

and

$$\begin{cases} \dot{x}(t) = f(x(t), c_\varepsilon(t)) \;, \forall t \in [t_a, t_b] \;, \\ x(t_a) = a + \varepsilon, \;\;\; x(t_b) = b, \;\;\; x(t) \geq 0, \forall t \in [t_a, t_b] \; . \end{cases}$$

The length of the statement of this assumption should not give the reader a false impression of excessive limitation of the study domain. Essentially, Assumption 1 states that with a larger initial stock, at least as much consumption as before becomes feasible over an interval, even if at the end of this interval the final stock is unchanged. Under this assumption, we can give some information about the occurence of the poorest generations over the time line.

[9]We thank an anonymous referee for those observations.

Theorem 3 (Location of the poorest generations) *Let the pair* $\{c^{mbr}(.), x^{mbr}(.)\}$ *be a solution to the MBR problem. Assume that* $c^{mbr}(.)$ *is not constant and* $x^{mbr}(.)$ *is unique. If the stock* x *is a productive asset (Assumption 1) the following properties hold:*

1. *when* $x^{mbr}(.)$ *is non-constant and weakly-increasing over time, then the poorest generations cannot be at the end of the sequence,*

2. *when* $x^{mbr}(.)$ *is non-constant, and weakly-decreasing over time, then the poorest generations cannot be at the beginning of the sequence.*

Proof. Appendix C. ■

In the perspective of appraising the ability of the MBR path to capture the two-phase logic of the just savings principle, it is the first item in the above theorem, where the stock of the resource is weakly-increasing, that is relevant. A pattern where the stock increases over time opens the possibility for future generations to enjoy higher levels of consumption. But, *a priori*, a trajectory where the lowest levels of consumption are in the far future cannot be excluded either, for it is admissible. However, Theorem 3 establishes that this possibility does not characterize a solution to the MBR problem.

The next results are helpful for comparing the MBR path with the discounted utilitarian (DU) path.

Corollary 1 *Let* $\{c^{mbr}(.), x^{mbr}(.)\}$ *be the unique solution starting from some* x_0*. Suppose* $x^{mbr}(.)$ *is non-constant and weakly increasing. Then there exists a finite time* T *such that after time* T *the solution* $(x^{mbr}(.), c^{mbr}(.))$ *is the solution of the discounted utilitarian program*

$$\max_{c} \int_{T}^{\infty} u(c)e^{-rt}dt$$

s.t. $\dot{x} = f(x, c)$, $x(T) = x_T^{mbr}$, *with* $x \geq 0$ *and* $c \geq 0$. *In particular* $c^{mbr}(t) > \underline{c}^{mbr}$ *for all* $t \geq T$.

Proof. This result follows from Claim 1 of Theorem 3. The complete proof is in Appendix D. ■

The above result is a generalization of Proposition 3, item (*ii*), established in Alvarez-Cuadrado & Long (2009) for a specific renewable resource model. It is the information provided by Theorem 3 that makes this generalization possible.

Theorem 4 *Under the assumptions of Corollary 1, if* $f(x, c)$ *is concave in* (x, c) *and* $u(c)$ *is concave then after time* T *the time path* $c^{mbr}(.)$ *is weakly increasing provided that* $f_c < 0$ *and* $f_{xc} \geq 0$.

Proof. Appendix E. ■

The above theorem is not only indicative of the behavior of the optimal consumption after some date T. It will also prove useful to establish that, when x_0 is below the modified golden rule stock, the upper level of consumption under the MBR criterion is achieved at infinity, *i.e.* $\sup_t \left\{c^{mbr}(.)\right\} = \lim_{t\to\infty} c^{mbr}(t)$, and to compare the MBR path with the discounted utilitarian path.

In view of this, consider now the following definition.

Definition 1 *Let*

$$I\left(c\left(.\right)\right) = \sup_t \left\{c\left(.\right)\right\} - \inf_t \left\{c\left(.\right)\right\} = \overline{c} - \underline{c}$$

be the range of consumptions distribution in trajectory $c(.)$.

One may expect that inequality, as measured by $I(c(.))$, is lower under the MBR criterion than under the discounted utilitarian criterion. This conjecture has to be ascertained. Let $c^{du}(t)$ denote the optimal consumption path under the discounted utilitarian criterion. Clearly $\underline{c}^{mbr} \geq \underline{c}^{du}$, by construction. But what about $\overline{c}^{mbr}$ and $\overline{c}^{du}$?

Theorem 5 (Inequality) *Let the assumptions underlying Corollary 1 and Theorem 4 jointly hold. Let $c^{du}(.)$ be the solution to the discounted utilitarian program. Suppose $x^{mbr}(.)$ is weakly increasing. Then*

$$I\left(c^{mbr}\left(.\right)\right) \leq I\left(c^{du}\left(.\right)\right),$$

i.e. inequality among generations, as captured by $I(c(.))$, is lower under the MBR criterion than under the discounted utilitarian criterion.

Proof. Appendix F. ■

Admittedly, $I(c(.))$ is coarse indicator of inequality. But it is relevant here, when comparing $c^{mbr}(.)$ and $c^{du}(.)$, because, when $x^{mbr}(.)$ is weakly increasing, both trajectories share the same upper-level of consumption $\overline{c}$, as can be deduced from Corollary 1. Therefore, inequalities between generations that take place in the intervals $\overline{c} - \underline{c}^{mbr}$ and $\overline{c} - \underline{c}^{du}$ could be very different, but at least the poorest are less far from the richest under the MBR scenario. The MBR path outperforms the DU path in the perspective of capturing the rawlsian idea that inequalities can be justified when they benefit the most disadvantaged people.

Two important remarks are in order.

Firstly, one may wonder under what conditions a solution $\left\{c^{mbr}(.), x^{mbr}(.)\right\}$ to the MBR problem exists with $x^{mbr}(.)$ being unique as required for most results of this paper, in particular Theorem 1. Ideally, one would like to know the necessary and sufficient conditions for the existence of a unique solution in order to determine precisely the scope of validity of our results. But such an undertaking would be illusory. For any given minimum consumption $\underline{c}$, the problem is a standard optimal control problem. Necessary conditions for the existence (and unicity) of an optimal control are notoriously elusive. Eventually, one has to decipher the necessary conditions for a differential equation to have a (unique) solution, a question about which little is known without imposing more structure. Sufficient conditions are easier to explore[10], but are of course more restrictive than one would like, in particular those required for unicity where strict concavity requirements are usually called upon (see our Theorem 2). However, it is clear that the situations we describe in this paper do exist. A first example is the numerical exercise performed by Alvarez-Cuadrado & Long (2009, Section 5) where $U(c) = \ln c$ and $f(x, c) = rx(1 - x/K) - c$, with $r, K > 0$ two positive parameters. We also give another example in Appendix G where $U(c) = c$. Here strict concavity of the Langragian L definitely does not hold. Still a solution exists and it is unique.

Secondly, a solution to the MBR problem is generally not time consistent: simply put, if an optimal trajectory $\left\{c^{mbr}(.), x^{mbr}(.)\right\}$ is reconsidered at a future time, there is no guarantee that the decisions for the remaining time will still be optimal. Intertemporal choice criterions do not necessarily lead to time-consistent optimal plans. In the axiomatic literature, a condition of *stationarity* is sometimes imposed on intertemporal social ranking, in order to ensure time-consistency. It is well-known that combining stationarity with a condition of *independence*, and other technical conditions, leads to the discounted utilitarian criterion (Koopmans, 1960), hence a dictatorship of the present (DP). Conversely, well-known criterions, *e.g.* RDU (Zuber and Asheim, 2012), MBR and Chichilnisky's that rule out DP - a minimum procedural requirement of equity among generations - are also time inconsistent. However, the restricted use of Chichilnisky's criterion (Figuières and Tidball, 2012) can provide a time consistent solution under specific economic environments. And Ramsey's criterion discards DP but it is time consistent, though it does not always return a number unless under special assumptions. It would be welcome to have a general picture of compatibilities and incompatibilities between intergenerational equity conditions and time consistency. Recently, Asheim and Mitra (2016) have proposed such an analysis. Time consistency was the central concern in a more specific context (see Asheim,1988), where technology includes both man-made capital and a non-renewable resource, and it was found that "time-inconsistency reappears in this technology" (p. 469). A solution to this intergenerational conflict would be to search for subgame perfect equilibrium where each generation chooses a strategy that is a best response to the strategies of later generations, as in Asheim (1988), where each generation's payoff is "the infinimum over the altruistic utility of all remaining generations" (p. 470). It would be interesting to formulate a game where each generation's payoff is the highest value of its MBR criterion. This is however beyond the scope of our paper, and is left for future research.

[10] Sufficient conditions for the existence of an optimal control in an inifinite horizon problem can be found for instance in Seierstad & Sydsæter (1987, Ch. 3, Theorem 15). For the existence of a solution to a differential equation, sufficient conditions are those of the well known Cauchy-Lipschitz Theorem.

5 A fully worked-out example and further results

This section studies an example with specific functional forms for the utility function and the dynamic equation. This will allows us to illustrate our general results and also to grasp a few additional insights.

The fundamentals are specified as follows:

$$\dot{x}(t) = f_0\left(x(t), c(t)\right) = ax(t) - c(t), \quad x\left(0\right) = x_0 \text{ given,}$$

for the stock equation, and:

$$U_0\left(c\right) = mc - \frac{n}{2}c^2 \ , \ m, n > 0,$$

for the standard of living. So there is a "bliss" level of consumption, $c^b = m/n$ and an upper bound on the value of the per period utility, $U^b = U\left(c^b\right)$.

The characterization of the discounted utilitarian solution is a direct application of *Pontryagin's Maximum Principle*. Details are left to the reader[11]. It can be shown that for $x(0) < m/(na)$, the optimal path is unique, and the state variable converges to the bliss stock steady state $x^b = m/(na)$[12]. The optimal consumption:

$$c^{du}(t) = c^b - (2a - \rho)(x^b - x_0)e^{(\rho - a)t} \ ,$$

increases towards the bliss level. In this accumulation context where future generations are also richer, the DU path is also the RDU consumption path; thus, more in tune with Rawls just savings principle, discounting here can be interpreted as an expression of inequality aversion rather than an intrinsic time preference. Notice that consumption, at any time, increases when the initial stock gets larger. The stock also increases and converges asymptotically to the bliss level x^b. This can be read[13] as a two-phase logic, an investment phase that increases the stock and a cruise phase that maintains it at the bliss level. This implies that the earlier poorer generations makes the sacrifice associated with accumulation, while the later richer generations are not required to do so.

Let us turn to the MBR program. Appendix H shows that the optimal consumption is continuous and not decreasing in t

$$c^{mbr}(t) = \begin{cases} \underline{c} & \text{if} \quad t \leq T \\ \\ c^b - \frac{\lambda(t)}{n(1-\theta)} & \text{if} \quad t \geq T. \end{cases}$$

where $\lambda(t)$ is the shadow price of the stock, a function that decreases toward 0 over time. As far as the stock is concerned, here also we have an investment

[11] They are available upon request.

[12] If $x(0) > m/na$, it will be optimal to consume at a constant level $c^{\#} = m/n$, and have strictly positive harvest surplus such that $x(t)$ will converge to the bliss stock steady state.

[13] Strictly speaking, the stock never ceases to increase at any finite time. But in the long run, increases becomes arbitrarily close to zero.

phase followed by a cruise phase, despite the initial phase of stagnation for consumption.

Therefore, both $c^{du}(t)$ and $c^{mbr}(t)$ converge to the bliss level of consumption, though the transitions are different. In contrast to the DU/RDU criterion, the MBR criterion leads to a weakly increasing path, because the role of the Rawlsian part is to protect the earlier poorer generations against excessive accumulation. In particular, there is an initial egalitarian phase. And, regarding this initial egalitarian phase, one can add:

Proposition 6 *Let the fundamental be f_0 and U_0. Assume the initial stock is below the bliss level, $x_0 < x^b$.*

1. *A greater initial condition $x_0 < x^b$ results in a greater consumption $\underline{c}$ for the poorest generations:*
$$\frac{d\underline{c}}{dx_0} > 0 \ .$$
2. *When the weight θ on the rawlsian part of MBR increases, the poor generations are better-off:*
$$\frac{d\underline{c}}{d\theta} > 0 \ .$$
3. *When the weight θ on the rawlsian part of MBR increases, the length T of the initial egalitarian phase increases:*
$$\frac{dT}{d\theta} > 0 \ .$$

A possible interpretation of these results is as follows: starting from a low value of θ, close to zero, the DU, RDU and the MBR paths are (nearly) the same. As θ increases, an initial egalitarian phase appears in the MBR program (and the higher θ the longer this phase). Intertemporal reallocations occur so that the accumulation effort is less demanding for the first poor generations.

6 Conclusion

This paper provides general theorems in respect of the control that maximizes the MBR intertemporal choice function. The main results are as follows: *i*) the state variable is shown to be monotonic under rather weak conditions, *ii*) we establish sufficient concavity conditions for a candidate trajectory to be optimal and unique, *iii*) we prove that inequality among generations, captured by the gap between the poorest and the richest generations, is lower when optimization is performed under the MBR criterion rather than under the discounted utilitarian criterion, *iv*) and, within a quadratic example, we provide an analytic solution to the MBR program in the case of scarcity of the resource ($x_0 < x^b$). This allows us to perform a sensitivity analysis with respect to some parameters of interest, the initial condition and the weight attributed to the less advantaged generations.

Those results are helpful to give a possible content to Rawls' (1971) just savings principle. The vagueness of this principle, in its implications, allows multiple interpretations. Several intertemporal social choice criteria, familiar to economists, could be contenders as incarnations of this principle. How are we to choose among them? According to Rawls, our final conception of justice should establish what he calls a "reflective equilibrium"—an acceptable balance between, on the one hand, deontological principles of justice and, on the other hand the consequences of applying those general principles to specific cases.

Following this methodology, we could then scrutinize various social welfare criteria and check, from a consequentialist point of view, their compliance to at least two aspects of the just saving principle: *i*) the necessity of a take-off phase if the initial conditions are two low, *ii*) a version of his difference principle, adjusted for the intertemporal context.

It is well know that the *maximin* criterion, sometimes erroneously attributed to Rawls, violates condition *i*). And Rawls also rejected the undiscounted utilitarian criterion. This criterion can be consistent with point *i*) but, by construction, it generally demands exorbitant sacrifices to the first generations, therefore it does not comply with point *ii*). From this perspective, it might be tempting argued that the practice of discounting future advantages, prescribed under Koopmans' logic (1960), is not so unfair after all. This is so because productive investment features an in-built bias in favour of the future that can be redressed by granting more importance to earlier generations. This is a standard argument, sometimes supported, sometimes challenged by economists who are working on climate change (see for instance the synthesis given by Dasgupta, 2008, or the critical assessment of Roemer, 2011, in particular Section 3 of his paper, or Section 5.1 in Asheim, 2010). Although it appears unfair from a deontological perspective (a *dictatorship of the present* in the words of Chichilnisky, 1996), when the optimal solution turns out to be among the subset of non-decreasing paths, the DU/RDU expresses an aversion for inequalities, rather than a preference for the present, and might be a not so bad candidate to embody the just savings principle. Our comparison of the optimal trajectories under the DU/RDU and the MBR criterion (Theorem 5 and the example of Section 5) simply shows that a special attention for the poor can be further added.

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A Proof of Theorem 1

The proof is by contradiction. Assume that $x^{mbr}(.)$ is not monotonic. Then, there exists a date $\tau \in [0, +\infty[$ and a strictly positive number θ such that:

$$x^{mbr}(\tau) = x^{mbr}(\tau + \theta),$$

with $x^{mbr}(.)$ not constant on the interval $[\tau, \tau + \theta]$.

Now define a new admissible path $\{\widetilde{c}(.), \widetilde{x}(.)\}$ with control and stable variables constructed as follows:

$$\begin{cases} \widetilde{c}(t) = c^{mbr}(t), \quad \widetilde{x}(t) = x^{mbr}(t), \quad \forall t \in [0, \tau], \\ \widetilde{c}(t) = c^{mbr}(t+\theta), \quad \widetilde{x}(t) = x^{mbr}(t+\theta), \quad \forall t \in]\tau, +\infty[. \end{cases} \tag{5}$$

By construction the pair $\{\widetilde{c}(.), \widetilde{x}(.)\}$ is admissible. And $\widetilde{c}(.)$ can also be made different from $c^{mbr}(.)$ over the interval $]\tau, +\infty[$. Indeed, if on the contrary $\widetilde{c}(t) = c^{mbr}(t+\theta) = c^{mbr}(t)$, $\forall t \in]\tau, +\infty[$, then by definition $c^{mbr}(t)$ is periodic - and not constant by assumption - on $]\tau, +\infty[$. In such a case one can simply choose two alternative numbers τ', θ' such that $[\tau', \tau' + \theta'] \sqsubset [\tau, \tau + \theta]$ and construct the above alternative path $\{\widetilde{c}(.), \widetilde{x}(.)\}$ using date τ' and θ' instead of τ and θ. Clearly $\{\widetilde{c}(.), \widetilde{x}(.)\}$ is different from $\left\{c^{mbr}(.), x^{mbr}(.)\right\}$ on the interval $]\tau', +\infty[$. To simplify, let us just consider that $\{\widetilde{c}(.), \widetilde{x}(.)\}$ is different from $\left\{c^{mbr}(.), x^{mbr}(.)\right\}$ on the interval $]\tau, +\infty[$.

Also, by construction,

$$\underline{c}^{mbr} = \inf_t \left\{ c^{mbr}(.) \right\} \leq \underline{\widetilde{c}} = \inf_t \left\{ \widetilde{c}(.) \right\} \ ,$$

or equivalently, using obvious notations

$$\underline{U}^{mbr} \leq \underline{\widetilde{U}} \ .$$

By definition $c^{mbr}(.)$ is the unique optimal solution, and necessarily:

$$W^{mbr}\left(c^{mbr}(.)\right) > W^{mbr}\left(\widetilde{c}(.)\right),$$

or,

$$\theta \underline{U}^{mbr} + (1-\theta) \int_{t=0}^{\infty} e^{-rt} U\left(c^{mbr}(t)\right) dt > \theta \underline{\widetilde{U}} + (1-\theta) \int_{t=0}^{\infty} e^{-rt} U\left(\widetilde{c}(t)\right) dt.$$

Hence,

$$\begin{aligned} \int_{t=0}^{\infty} e^{-rt} U\left(c^{mbr}(t)\right) dt &> \int_{t=0}^{\infty} e^{-rt} U\left(\widetilde{c}(t)\right) dt \ , \\ \int_{t=\tau}^{\infty} e^{-rt} U\left(c^{mbr}(t)\right) dt &> \int_{t=\tau}^{\infty} e^{-rt} U\left(\widetilde{c}(t)\right) dt \ , \\ \int_{t=\tau}^{\infty} e^{-rt} U\left(c^{mbr}(t)\right) dt &> \int_{t=\tau}^{\infty} e^{-rt} U\left(c^{mbr}(t+\theta)\right) dt, \end{aligned} \tag{6}$$

where the second line obtains because the two controls coincide until date τ, and the last line makes use of (5). Notice that:

$$\int_{t=\tau}^{\infty} e^{-rt} U\left(c^{mbr}(t+\theta)\right) dt = e^{r\theta} \int_{t=\tau+\theta}^{\infty} e^{-rt} U\left(c^{mbr}(t)\right) dt.$$

With this expression, inequality (6) can be written:

$$\int_{t=\tau}^{\infty} e^{-rt} U\left(c^{mbr}(t)\right) dt > e^{r\theta} \int_{t=\tau+\theta}^{\infty} e^{-rt} U\left(c^{mbr}(t)\right) dt,$$

consequently:

$$\int_{t=\tau}^{\tau+\theta} e^{-rt} U\left(c^{mbr}(t)\right) dt > \left(e^{r\theta} - 1\right) \int_{t=\tau+\theta}^{\infty} e^{-rt} U\left(c^{mbr}(t)\right) dt. \tag{7}$$

Next define a new admissible path $\{\overline{c}(.),\overline{x}(.)\}$ with control and stable variable now constructed as follows:

$$\begin{cases} \overline{c}(t) = c^{mbr}(t),\ \overline{x}(t) = x^{mbr}(t), \quad \forall t \in [0,\tau+\theta], \\ \overline{c}(t) = c^{mbr}(t-\theta),\ \ \overline{x}(t) = x^{mbr}(t-\theta), \quad \forall t \in \,]\tau+\theta,+\infty[\,. \end{cases} \tag{8}$$

Again, by construction the pair $\{\overline{c}(.),\overline{x}(.)\}$ is admissible and different from $\left\{c^{mbr}(.),x^{mbr}(.)\right\}$. Also by construction,

$$\underline{c}^{mbr} = \inf_t \left\{c^{mbr}(.)\right\} = \underline{\overline{c}} = \inf_t \left\{\overline{c}(.)\right\} \ ,$$

or

$$\underline{U}^{mbr} = \underline{\overline{U}} \ . \tag{9}$$

We now compare the value of $W^{mbr}(.)$ for $\{\overline{c}(.),\overline{x}(.)\}$ and $\left\{c^{mbr}(.),x^{mbr}(.)\right\}$. Using definition (8), equality (9) and inequality (7):

$$\begin{aligned} W^{mbr}(\overline{c}(.)) - W^{mbr}(c^{mbr}(.)) &= \int\limits_{t=0}^{\infty} e^{-rt}U(\overline{c}(t))\,dt - \int\limits_{t=0}^{\infty} e^{-rt}U\left(c^{mbr}(t)\right)dt \\ &= \int\limits_{t=\tau+\theta}^{\infty} e^{-rt}U\left(c^{mbr}(t-\theta)\right)dt - \int\limits_{t=\tau+\theta}^{\infty} e^{-rt}U\left(c^{mbr}(t)\right)dt \\ &= e^{-r\theta}\int\limits_{t=\tau}^{\infty} e^{-rt}U\left(c^{mbr}(t)\right)dt - \int\limits_{t=\tau+\theta}^{\infty} e^{-rt}U\left(c^{mbr}(t)\right)dt \\ &= e^{-r\theta}\left[\int\limits_{t=\tau}^{\tau+\theta} e^{-rt}U\left(c^{mbr}(t)\right)dt - \left(e^{r\theta}-1\right)\int\limits_{t=\tau+\theta}^{\infty} e^{-rt}U\left(c^{mbr}(t)\right)dt\right] > 0, \end{aligned}$$

a contradiction.

A quicker way to understand and heuristically prove this theorem is as follows[14]. The first part of the proof establishes that the average DU payoff on $[\tau,\tau+\theta]$ is higher than the average DU payoff on $[0,\infty)$. (Because otherwise, one could have done as well by skipping $[\tau,\tau+\theta]$ which contradicts uniqueness.) But then one can improve the path by repeating the behavior on $[\tau,\tau+\theta]$, leading to a contradiction that is established in the second part of the proof.

B Necessary conditions, sufficient conditions and uniqueness

B.1 Necessary conditions

First, recall the following identity

$$1 = \int_0^{\infty} re^{-rt}dt \ .$$

[14]We thank an anonymous referee for this intuition.

Thus, a solution to the MBR problem is a triple $\left(c^{mbr}(.), x^{mbr}(.), \underline{c}^{mbr}\right)$ that maximizes

$$\int_0^\infty e^{-rt} \left[r\theta U(\underline{c}) + (1-\theta)U(c(t))\right] \ dt \ ,$$

subject to

$$\dot{x} = f(x\,(t)\,, c\,(t)) \ ,$$
$$c(t) - \underline{c} \geq 0 \ .$$

Following the approach of Hestenes[15], we treat $\underline{c}$ as a "control parameter", *i.e.* a variable that,once chosen, remains constant over the time horizon $[0, \infty[$. We define the Lagragian:

$$L = e^{-rt} \left[r\theta U(\underline{c}) + (1-\theta)U(c(t))\right] + \psi(t) f(x\,(t)\,, c\,(t)) + \lambda(t)\left(c(t) - \underline{c}\right)$$

The necessary conditions are:

$$\frac{\partial L}{\partial c} = 0 \ ,$$
$$\dot{\psi} = -\frac{\partial L}{\partial x} \ ,$$
$$\dot{x} = \frac{\partial L}{\partial \psi} \ ,$$
$$\lambda(t) \geq 0, \ c(t) - \underline{c} \geq 0, \ \lambda(t)\left(c(t) - \underline{c}\right) = 0 \ ,$$
$$\frac{\partial}{\partial \underline{c}} \int_0^\infty e^{-rt} \left[r\theta U(\underline{c}) + (1-\theta)U(c(t))\right] dt + \frac{\partial}{\partial \underline{c}} \int_0^\infty \lambda(t)\left(c(t) - \underline{c}\right) dt = 0 \ . \quad (10)$$

The latter condition reduces to

$$\theta U'(\underline{c}) - \int_0^\infty \lambda(t) dt = 0$$

Adn the transversality conditions are:

$$\lim_{t\to\infty} \psi(t) \quad \geq \quad 0, \quad (11)$$
$$\lim_{t\to\infty} \psi(t) x\,(t) \quad = \quad 0 \ . \quad (12)$$

B.2 Sufficient conditions and uniqueness

Our proof is similar to that of Takayama (1986).

For simplicity, we use the following notations

$$L^{mbr} = L(c^{mbr}, x^{mbr}, \underline{c}^{mbr}, \psi^{mbr}, \lambda^{mbr}, t) \ ,$$

and

$$L^{\#} = L(c^{\#}, x^{\#}, \underline{c}^{\#}, \psi^{mbr}, \lambda^{mbr}, t) \ , \quad \text{(note: } \psi^{mbr}, \lambda^{mbr} \text{ are not typos here)},$$

[15] See, for example, Takayama, A. (1986), *Mathematical Economics*, second edition, Cambridge University Press, Cambridge and New York

where the "mbr" over the multipliers indicates that we use the same path $\left(\psi^{mbr}(.), \lambda^{mbr}(.)\right)$ for both L^{mbr} and $L^{\#}$.

Since $\lambda^{mbr}\left[c^{mbr} - \underline{c}^{mbr}\right] = 0$,

$$V^{mbr} = \int_0^{\infty} \left[L^{mbr} - \psi^{mbr}\dot{x}^{mbr}\right] dt$$

Now, since $\lambda^{mbr} \geq 0$ and since, by feasibility, $c^{\#} - \underline{c}^{\#} \geq 0$, we have $\lambda^{mbr}\left(c^{\#} - \underline{c}^{\#}\right) \geq 0$, hence

$$V^{\#} = \int_0^{\infty} \left[L^{\#} - \psi^{mbr}\dot{x}^{\#} - \lambda^{mbr}\left(c^{\#} - \underline{c}^{\#}\right)\right] dt \leq \int_0^{\infty} \left[L^{\#} - \psi^{mbr}\dot{x}^{\#}\right] dt$$

Then

$$\begin{aligned} V^{mbr} - V^{\#} \quad &\geq \quad -\int_0^{\infty} \left[\psi^{mbr}\dot{x}^{mbr} - \psi^{mbr}\dot{x}^{\#}\right] dt \\ &\quad + \int_0^{\infty} \left[L^{mbr} - L^{\#}\right] dt \end{aligned}$$

Now, under the assumption that L is concave

$$L^{mbr} - L^{\#} \geq \left(x^{mbr} - x^{\#}\right)\frac{\partial L^{mbr}}{\partial x} + \left(c^{mbr} - c^{\#}\right)\frac{\partial L^{mbr}}{\partial c} + \left(\underline{c}^{mbr} - \underline{c}^{\#}\right)\frac{\partial L^{mbr}}{\partial \underline{c}}$$

with **strict inequality** if L is strictly concave.

Now from the necessary conditions $\frac{\partial L^{mbr}}{\partial c} = 0$, $\frac{\partial L^{mbr}}{\partial x} = -\dot{\psi}^{mbr}$.Then

$$L^{mbr} - L^{\#} \geq -\dot{\psi}^{mbr}(x^{mbr} - x^{\#}) + \left(\underline{c}^{mbr} - \underline{c}^{\#}\right)\frac{\partial L^{mbr}}{\partial \underline{c}}$$

Therefore

$$V^{mbr} - V^{\#} \geq -\int_0^{\infty} \left[\dot{\psi}^{mbr}(x^{mbr} - x^{\#}) + \psi^{mbr}\dot{x}^{mbr} - \psi^{mbr}\dot{x}^{\#}\right] dt +$$

$$+ \left(\underline{c}^{mbr} - \underline{c}^{\#}\right)\int_0^{\infty} \left[\frac{\partial L^{mbr}}{\partial \underline{c}}\right] dt$$

Since $\int_0^{\infty}\left[\frac{\partial L^{mbr}}{\partial \underline{c}}\right] dt = 0$ by condition (10), we obtain

$$V^{mbr} - V^{\#} \geq -\lim_{t\to\infty}\left[\psi^{mbr}(t)x^{mbr}(t) - \psi^{mbr}(0)x^{mbr}(0)\right] + \lim_{t\to\infty}\left[\psi^{mbr}(t)x^{\#}(t) - \psi^{mbr}(0)x^{\#}(0)\right]$$

Using the fixed initial condition, $x_0^{mbr} = x_0^{\#} = x_0$, the above inequality becomes

$$V^{mbr} - V^{\#} \geq \lim_{t\to\infty} \psi^{mbr}(t)\left[x^{\#}(t) - x^{mbr}(t)\right]$$

which is positive as can be deduced from the transversality conditions (11) and (12) and because $x^{\#}(t) \geq 0$. With a strictly concave L, we obtain uniqueness.

C Proof of Theorem 3

Claim 1. Recall that, if $c^{mbr}(.)$ and $x^{mbr}(.)$ are a solution to the MBR problem where $c^{mbr}(.)$ is not constant and $x^{mbr}(.)$ is unique, then $x^{mbr}(.)$ is monotonic (by Theorem 1).

In order to prove Claim 1, assume on the contrary that $x^{mbr}(.)$ is non-constant and weakly increasing over time but the poorest generation(s) is/are at the end of the sequence. Let $\bar{t} > 0$ be the earliest date at which the lowest level of consumption is achieved. That is,

$$\bar{t} \equiv \inf_{t} \left\{ t : c^{mbr}(t') = \underline{c}^{mbr}, \forall t' \geq t \right\}$$

Thus, after $\bar{t}$, the consumption path is constant.

Then, there exists a number $d > 0$ such that $c^{mbr}\left(\bar{t} - d\right) > c^{mbr}\left(\bar{t}\right) = \underline{c}^{mbr}$ and $c^{mbr}\left(\bar{t} - d'\right) > \underline{c}^{mbr}$ for all $d' \in (0, d)$.

At time $\bar{t} - d$, the stock is $x^{mbr}(\bar{t} - d) \leq x^{mbr}(\bar{t})$. Since the stock $x^{mbr}(\bar{t} - d)$ can sustain a stream of consumption with an initial phase of length d with $c > c^{mbr}\left(\bar{t}\right)$ followed by a phase of constant consumption $\underline{c}^{mbr}$, it follows from Assumption 1 that starting from time $\bar{t}$ with stock level $x^{mbr}(\bar{t}) \geq x^{mbr}(\bar{t} - d)$, it is possible to sustain a stream of consumption c^{**} with an initial phase $\left[\bar{t}, \bar{t} + d\right]$ such that $c^{**}(t) > c^{mbr}(t)$ and for all $t \in \left[\bar{t}, \bar{t} + d\right]$, and $c^{**}(t) = \underline{c}^{mbr}$ for all $t > \bar{t} + d$.

To summarize, the following alternative sequence $c^{**}(.)$ is admissible:

$$\left\{ \begin{array}{c} c^{**}(t) = c^{mbr}(t), \; x^{**}(t) = x^{mbr}(t), \quad \forall t \in \left[0, \bar{t}\right[, \\ c^{**}(t) \geq c^{mbr}(t - d), \quad \forall t \in \left[\bar{t}, +\infty\right[\text{ , with equality for all } t \in \left[\bar{t} + d, +\infty\right[\end{array} \right. \tag{13}$$

and, by construction, consumptions under the two possibilities are identical except over the interval $\left[\bar{t}, \bar{t} + d\right]$ where one has $c^{**}(t) > c^{mbr}(t)$.

Comparing the value of the MBR criterion under the optimal path and the alternative path, one has:

$$\begin{aligned} W^{mbr}(c^{mbr}(.)) - W^{mbr}(c^{**}(.)) &= \int_{t=0}^{\infty} e^{-rt} U\left(c^{mbr}(t)\right) dt - \int_{t=0}^{\infty} e^{-rt} U\left(c^{**}(t)\right) dt \ , \\ &\leq \int_{t=\bar{t}}^{\bar{t}+d} e^{-rt} \underbrace{\left[U\left(c^{mbr}(t)\right) - U\left(c^{*}(t-d)\right)\right]}_{<0} dt \ < 0 \ , \end{aligned}$$

a contradiction.

Claim 2. The proof follows a logic similar to that of Claim 1. Assume on the contrary that $x^{mbr}(.)$ is non-constant and weakly decreasing over time, but the poorest generation(s) is/are at the beginning of the sequence. Suppose there exists an initial interval $\left[0, \bar{t}\right]$, with $c^{mbr}(t) = \underline{c}^{mbr}$, and there exists $\delta > 0$ such that $c^{mbr}(t) > c^{mbr}\left(\bar{t}\right), \forall t \in \left(\bar{t}, \bar{t} + \delta\right)$, and $c^{mbr}(t) \geq c^{mbr}(\bar{t})$ for all $t \geq \bar{t} + \delta$.

Since $x^{mbr}(0) \geq x^{mbr}(\bar{t})$, it follows from Assumption 1 that we can construct a time path $c^{**}(t)$ such that $c^{**}(t) = c^{mbr}(t + \bar{t})$, for all $t \geq 0$. Comparing the value of the MBR criterion under the optimal path and the alternative path, one has:

$$\begin{aligned} W^{mbr}(c^{mbr}(.)) - W^{mbr}(c^{**}(.)) &= \int_{t=0}^{\infty} e^{-rt}\{U\left(c^{mbr}(t)\right) - \underline{U}\}dt - \int_{t=0}^{\infty} e^{-rt}\{U\left(c^{**}(t)\right) - \underline{U}\}dt \ < 0 \\ &= \int_{t=\bar{t}}^{\infty} e^{-rt}\{U\left(c^{mbr}(t)\right) - \underline{U}\}dt - \int_{\tau=0}^{\infty} e^{-r\tau}\{U\left(c^{mbr}(\tau+\bar{t})\right) - \underline{U}\}d\tau \\ &= \int_{t=\bar{t}}^{\infty} e^{-rt}\{U\left(c^{mbr}(t)\right) - \underline{U}\}dt - \int_{t=\bar{t}}^{\infty} e^{-r(t-\bar{t})}\{U\left(c^{mbr}(t)\right) - \underline{U}\}dt < 0 \end{aligned}$$

a contradiction.

D Proof of Corollary 1

Recall the necessary conditions of optimality for the MBR problem given in appendix B. They are:

$$i) \quad \frac{\partial L}{\partial c} = 0 \ ,$$

$$\Leftrightarrow \quad e^{-rt}(1-\theta)U'(c(t)) + \psi(t)f_2(x\left(t\right), c\left(t\right)) + \lambda(t) = 0$$

$$ii) \quad \dot{\psi} = -\frac{\partial L}{\partial x} \ ,$$

$$\Leftrightarrow \quad \dot{\psi} = -\psi(t)f_1(x\left(t\right), c\left(t\right))$$

$$iii) \quad \dot{x} = \frac{\partial L}{\partial \psi} \ ,$$

$$\Leftrightarrow \quad \dot{x} = f(x\left(t\right), c\left(t\right))$$

$$iv) \quad \lambda(t) \geq 0, \ c(t) - \underline{c} \geq 0, \ \lambda(t)\left(c(t) - \underline{c}\right) = 0$$

$$v) \quad \frac{\partial}{\partial \underline{c}} \int_0^{\infty} e^{-rt}\left[r\theta U(\underline{c}) + (1-\theta)U(c(t))\right]dt + \frac{\partial}{\partial \underline{c}} \int_0^{\infty} \lambda(t)\left(c(t) - \underline{c}\right)dt = 0 \tag{14}$$

The latter condition reduces to

$$\theta U'(\underline{c}) - \int_0^{\infty} \lambda(t)dt = 0$$

Now, consider that $c(t) > \underline{c}$, $\forall t \geq T$, (and the stock achieved at date T is $x\left(T\right) = x_T^{mbr}$). From condition $iv)$ one can deduce $\lambda(t) = 0, \forall t \geq T$, and the necessary conditions boils down to:

$$\begin{aligned} e^{-rt}(1-\theta)U'(c(t)) + \psi(t)f_2(x\left(t\right), c\left(t\right)) &= 0 \\ \dot{\psi} &= -\psi(t)f_1(x\left(t\right), c\left(t\right)) \\ \dot{x} &= f(x\left(t\right), c\left(t\right)) \\ \theta U'(\underline{c}) &= \int_0^{T} \lambda(t)dt \end{aligned}$$

Define the new variable $\sigma(t) = \psi(t)/(1-\theta)$. Then the three first conditions above can be rewritten as:

$$\begin{aligned} e^{-rt}U'(c(t)) + \sigma(t) f_2(x(t), c(t)) &= 0 \ , \\ \dot{\sigma} &= -\sigma(t) f_1(x(t), c(t)) \ , \\ \dot{x} &= f(x(t), c(t)) \ , \end{aligned}$$

$\forall t \geq T$. The proof is completed once one observes that these expressions are the necessary conditions associated to the discounted utilitarian program that starts at date T with initial condition $x(T) = x_T^{mbr}$, and where the Hamiltonian is:

$$H = e^{-rt}U(c(t)) + \sigma(t) f(x(t), c(t)).$$

E Proof of Theorem 4

From date T the program is utilitarian, and the value function $V(x)$ is concave because $f(x.c)$ is concave in (x, c) and $U(c)$ is concave.[16] Also, because $V(x)$ is concave, then it is continuous and differentiable almost everywhere (see Nicolescu & Person, 2006), that is the set of points x where the left hand and the right hand derivatives of $V(x)$, which we can denote by $V_L'(x)$ and $V_R'(x)$, are different is at most countable.

Take any two points of time, t_1 and t_2, such that $t_2 > t_1$. Then $x(t_2) \geq x(t_1)$ (by assumption). First we analyze the case where $V(.)$ is differentiable. And after we consider the case where the right hand and left hand derivative of $V(.)$ do not coïncide.

Because $x(t_2) \geq x(t_1)$, then $V'(x_2) \leq V'(x_1)$ (by concavity of $V(.)$). We now show that $c(t_2) \geq c(t_1)$. For simplicity of notation, we write x_i and c_i for $x(t_i)$ and $c(t_i)$, for $i = 1, 2$.

The HJB equation is

$$rV(x) = \max_c \left[U(c) + V'(x)f(x, c)\right] \ .$$

Then, the first order condition for the right-hand-side is:

$$U'(c) + V'(x)f_c(x, c) = 0 \ . \tag{15}$$

Therefore:

$$U'(c_2) = -f_c(x_2, c_2)V'(x_2) \ ,$$

and

$$\begin{aligned} U'(c_1) &= -f_c(x_1, c_1)V'(x_1) \\ &= -f_c(x_2, c_2)V'(x_1) + V'(x_1)\left[f_c(x_2, c_2) - f_c(x_1, c_1)\right] \\ &= -f_c(x_2, c_2)V'(x_1) + V'(x_1)\left\{\left[f_c(x_2, c_2) - f_c(x_1, c_2)\right] + \left[f_c(x_1, c_2) - f_c(x_1, c_1)\right]\right\} \end{aligned}$$

[16] See Long (1979) for a proof of the concavity of $V(x)$.

Then

$$[U'(c_1) - U'(c_2)] + V'(x_1)\,[f_c(x_1, c_1) - f_c(x_1, c_2)] =$$
$$-f_c(x_2, c_2)\,[V'(x_1) - V'(x_2)] + V'(x_1)\,[f_c(x_2, c_2) - f_c(x_1, c_2)] \qquad (16)$$

The right-hand side is positive or zero, because $f_c(x_2, c_2) - f_c(x_1, c_2) \geq 0$ (this follows from $f_{xc} \geq 0$). Therefore the left hand side must be positive or zero. This implies that $c_2 \geq c_1$. (Suppose $c_2 < c_1$; then the left hand side would be negative, since the functions U' and f_c are decreasing in c; therefore we would have a contradiction).

In order to complete the proof, let us see what happens if V is not differentiable at x_1 or at x_2? Then in equation (15), $V'(x)$ may correspond to the left hand or the right hand derivative, $V'_L(x)$ and $V'_R(x)$. The proof is still valid, because if $x_2 \geq x_1$ then the concavity of V implies

$$\min\{V'_L(x_1), V'_R(x_1)\} \geq \max\{V'_L(x_2), V'_R(x_2)\}$$

Then the RHS of (16) is still positive or zero, regardlless of which derivatives we used. Any two paths of $c\,(.)$ that differ from each other at isolated points in time are essentially identical, and the resulting path of the state variable is not affected.

F Proof of Theorem 5

First observe that

$$\lim_{t\to\infty} c^{du}(t) = \bar{c}^{du} \ ,$$

because, under the assumptions of Theorem 4, consumption is weakly increasing over time. And, as usual, the value of the steady state does not depend on the initial condition.

Observe also that, by virtue of Corollary 1:

$$\lim_{t\to\infty} c^{mbr}(t) = \bar{c}^{du}.$$

We can also establish that

$$\lim_{t\to\infty} c^{mbr}(t) = \bar{c}^{mbr} \ .$$

Indeed assume on the contrary that the more advantaged generations occur in finite time at some date T'. Necessarily this date occurs before the date T at which the MBR trajectory has increasing consumptions (Theorem 4). Then $c^{mbr}(T') > \lim_{t\to\infty} c^{mbr}(t)$ and there exists a number d such that:

$$c^{mbr}(t) \geq \lim_{t\to\infty} c^{mbr}(t) \quad \forall t \in [T', T' + d[\ .$$

Then consider the alternative consumption trajectory:

$$\begin{cases} \widehat{c}\,(t) = c^{mbr}(t),\ \forall t \in [0, T[\,, \\ \widehat{c}\,(t) = c^{mbr}(T' + t),\ \forall t \in [T, T + d[\,, \\ \widehat{c}\,(t) = c^{mbr}(t),\ \forall t \in [T + d, +\infty[\,. \end{cases}$$

This trajectory is admissible and has the same consumptions as the one that maximizes the MBR criterion, except over a finite interval where generations enjoy a higher consumption than under the MBR solution, a contradiction. Therefore we have established $\lim_{t\to\infty} c^{mbr}\,(t) = \overline{c}^{mbr}$. To summarize:

$$\overline{c}^{mbr} = \overline{c}^{du}\ .$$

Since by definition it is also true that $\underline{c}^{mbr} \geq \underline{c}^{du}$, necessarily:

$$I\left(c^{mbr}\,(.)\right) = \overline{c}^{mbr} - \underline{c}^{mbr} \leq I\left(c^{du}\,(.)\right) = \overline{c}^{du} - \underline{c}^{du}\ .$$

QED.

G A simple MBR problem without strict concavity of the Lagrangian

Here we present a simple example without strict concavity of the Lagrangian. Yet the MBR solution exists and is unique.

A MBR problem can be solved in two steps. In the first step, for any given minimum consumption $\underline{c}$, the problem is a standard optimal control problem, for which one can find a solution configured by $\underline{c}$. The second step is to plug this solution in the MBR criterion, call $V(\underline{c})$ the resulting value function, and proceed with the maximization of $V(\underline{c})$ with respect to $\underline{c}$. In the second step one has to cope with a static maximization problem.

Assume the dynamics are:

$$\dot{x}\,(t) = G(x\,(t)) - c\,(t)\,, \quad x(0) = x_0, \tag{17}$$

where $0 \leq \underline{c} < c\,(t) < \overline{c}$. We also assume that $\max_x G\,(x) < \overline{c}$. And let the instantaneous utility function be:

$$U(c) = c\,.$$

In the first step, the problem is to

$$max_{c(.)} \int_0^\infty e^{-\rho t}\ c\,(t)\ dt, \tag{18}$$

subject to (17).

Denote x_u^* the solution to the Euler equation $G'(x) = \rho$. From the Turnpike Theorem (see for instance Clark, 1990, Ch. 2, page 53) one can deduce that, if $\underline{c} < G(x_u^*) < \overline{c}$, the solution $c^*(t)$ to (18) is:

$$c^*(t) = \begin{cases} \overline{c} & \text{if} \quad x(t) > x_u^* \\ \underline{c} & \text{if} \quad x(t) < x_u^* \\ G(x_u^*) & \text{if} \quad x(t) = x_u^* \end{cases} \tag{19}$$

Note that some further restrictions on $\underline{c}$ must be imposed for the above program to be a solution. Indeed, $c^*(t)$ can only take the three values given above and, if at a prevailing value of $x < x_u^*$, $\underline{c}$ is too large in the sense $\underline{c} > G(x)$, then $\dot{x} < 0$ which means that, to remain in the admissible domain of x - made of non negative real numbers - at some point in time T, the control should switch to $c^*(T) = 0$, a value that does not fall within the set of admissible values $0 \leq \underline{c} < c(t) < \overline{c}$. In order to avoid this, we impose that $\underline{c} < G(x_0)$.

Now let us move to the second step. For a given and constant c, let us call $x^c(t)$ the solution to $\dot{x} = G(x) - c$. Depending on the value of the initial condition, two cases must be distinguished:

1. When $x_0 < x_u^*$, and since $\underline{c} < G(x_0)$, the value $V(\underline{c})$ associated to the MBR criterion is:

$$V(\underline{c}) = (1-\theta)\left(\int_0^{T^{\underline{c}}} e^{-\rho t}\ \underline{c}\ dt + \int_{T^{\underline{c}}}^{\infty} e^{-\rho t}\ G(x_u^*)\ dt\right) + \theta\ \underline{c}. \tag{20}$$

   where $T^{\underline{c}}$ is such that $x^{\underline{c}}(T^{\underline{c}}) = x_u^*$. Keep in mind that (20) gives the value of the program when $c = \underline{c}$ and $\dot{x}^{\underline{c}} > 0$ for $t \in [0, T^{\underline{c}}]$.
   Let us consider a numerical specification with $\rho = 0.01$, $G(x) = x(1-x)$. Under this specification $x_u^* = 0.495$, $G(x_u^*) = 0.249975$. Let us pick $x_0 = 0.1 < x_u^*$. For this initial stock, the condition $\underline{c} < G(x_0)$ becomes $\underline{c} \leq 0.09$. Therefore, the problem is just to pick $\underline{c} \in [0, 0.09]$ in order to maximize (20). According to (19) the optimal plan is first $c^*(t) = \underline{c}$ until date $T^{\underline{c}}$, where the decision switches to $c^*(t) = G(x_u^*)$. And $x^c(t)$ is increasing up to $x_u^* = 0.495$. This solution can be plugged into (20) in order to give $V(\underline{c})$. The next question is: what is the optimal value of $\underline{c}$? The answer depends on θ. Numerically, we can observe that if $\theta = 0.5$, $V(\underline{c})$ is decreasing and concave, with a maximum at $\underline{c} = 0$ (see Figure 1). But if $\theta = 0.8$, then $V(\underline{c})$ is just concave with a maximum at $\underline{c} = 0.056$ (see Figure 2).

2. When $x_0 > x_u^*$, the optimal control starts as $c^*(t) = \overline{c}$, and necessarily $\dot{x} < 0$ since by assumption $\overline{c} > \max_x G(x)$. Therefore the stock decreases down to x_u^*. Arrived at that point, the optimal control switches to $c^*(t) = G(x_u^*)$. And the value of the MBR criterion is:

$$V(G(x_u^*)) = (1-\theta)\left(\int_0^{T^{\overline{c}}} e^{-\rho t}\ \overline{c}\ dt + \int_{T^{\overline{c}}}^{\infty} e^{-\rho t}\ G(x_u^*)\ dt\right) + \theta\ G(x_u^*)\ ,$$

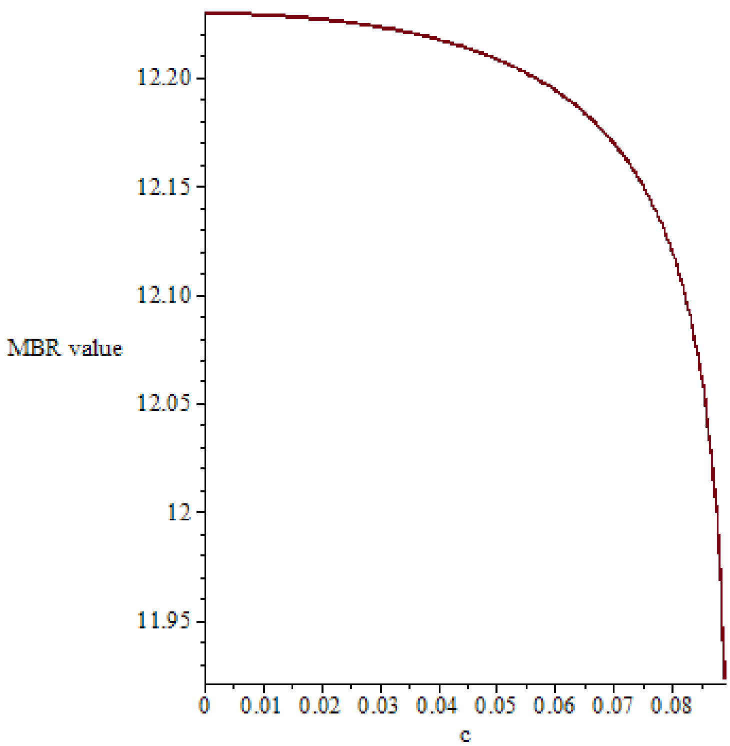

Figure 1: Value function for $\theta = 0.5$

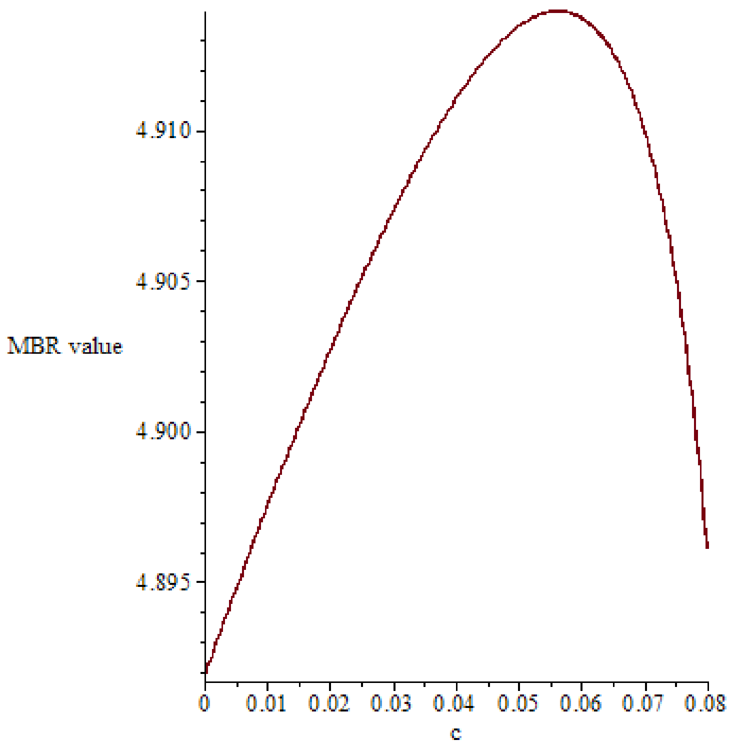

Figure 2: Value function for $\theta = 0.8$

where $T^{\overline{c}}$ is defined by $x^{\overline{c}}(T^{\overline{c}}) = x_u^*$. Notice that the poorest generations consume $c^*(t) = G(x_u^*)$, an exogenous value about which there is no optimal choice to be made in the second step, whatever the value taken by parameter θ.

H Characterization of the MBR optimal program in a quadratic example

The problem is

$$\max_{s(.),c(.),\underline{u}} \theta\underline{u} + (1-\theta)\int_0^\infty (mc - n/2c^2)e^{-\rho t}dt, \qquad 0 < \rho < a \ .$$

subject to the constraints:

$$\begin{aligned} \dot{x} &= ax - c \ , \\ x\left(0\right) &= x_0 < x^b = \frac{m}{na}, \\ x(t) &\geq 0 \text{ for all } t. \end{aligned}$$

The Hamiltonian is:

$$H = (1-\theta)(mc - \frac{n}{2}c^2) + \lambda\left(ax - c\right) + w_3(mc - \frac{n}{2}c^2 - \underline{u}) \ ,$$

Necessary conditions for optimality are:

$$\frac{\partial}{\partial c}H = (1-\theta+w_3)(m-nc) - \lambda = 0 \quad \Leftrightarrow \quad c = \frac{m}{n} - \frac{\lambda}{(1-\theta+w_3)n} \ , \tag{21}$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial}{\partial x}H = (\rho - a)\lambda \quad \Leftrightarrow \quad \lambda(t) = \lambda_0 e^{(\rho-a)t}, \tag{22}$$

$$\dot{x} = ax - \left(\frac{m}{n} - \frac{\lambda}{(1-\theta+w_3)n}\right) - s \ . \tag{23}$$

$$w_3 \geq 0, \quad w_3(mc - \frac{n}{2}c^2 - \underline{u}) = 0, \quad \theta = \int_{t=0}^\infty e^{-\rho t}w_3(t)dt \ . \tag{24}$$

Note that $w_3 = 0$ for all t is not possible because (24) is not verified.

For the case $x(0) < m/(na)$, we are going to prove that there exists $T > 0$ such that $c(t) = \underline{c} \in [0,T]$, with $\underline{c} < c^B = m/n$ verifying first order conditions. Using equation (21), we define:

$$w_3(t) = \begin{cases} \frac{\lambda(t)}{m-\underline{c}n} - (1-\theta) & \text{if} \quad t \leq T \\ 0 & \text{if} \quad t \geq T. \end{cases}$$

with T such that $w_3(T) = 0$:

$$\frac{\lambda(T)}{m - \underline{c}n} = (1 - \theta) \iff T = \frac{1}{a - \rho} ln \left(\frac{\lambda_0}{(1 - \theta)(m - \underline{c}n)} \right). \qquad (25)$$

Note that since $\rho < a$ asking for $T > 0$ implies that $\ln \left(\frac{(1-\theta)(m-\underline{c}n)}{\lambda_0} \right) > 0$ and that $\underline{c} < m/n \iff \lambda_0 > 0$.

We now need the following partial result:

Lemma 1 *Solution to the differential equation (23) is*

$$x(t) = \begin{cases} \frac{\underline{c}}{a} + e^{at}(x_0 - \frac{\underline{c}}{a}) & \text{if} \quad t \leq T \\ x^b - \frac{\lambda(t)}{(2a-\rho)n(1-\theta)} & \text{if} \quad t \geq T. \end{cases}$$

Proof. Available upon request. ■

We must find λ_0 and $\underline{c}$. We have two equations for this purpose. The first one comes from the above expression that gives the value of x at time T:

$$\frac{\underline{c}}{a} + e^{aT}(x_0 - \frac{\underline{c}}{a}) = x^b - \frac{\lambda(T)}{(2a - \rho)n(1 - \theta)} .$$

After substitution of T as indicated in (25) and rearrangements, it becomes

$$\left(\underline{c} - \frac{m}{n}\right) \left(\frac{\rho - a}{a(\rho - 2a)} \right) + y^{\frac{a}{a-\rho}} \left(x_0 - \frac{\underline{c}}{a} \right) = 0. \qquad (26)$$

where $y = \frac{\lambda_0}{(m-\underline{c}n)(1-\theta)}$.

The second equation is $\theta = \int_{t=0}^{\infty} e^{-\rho t} w_3(t) dt$, that after some arrangement becomes

$$\frac{y}{a} - \frac{(\rho - a)}{a\rho} y^{\frac{\rho}{\rho - a}} - \frac{\theta}{1 - \theta} - \frac{1}{\rho} = 0. \qquad (27)$$

We can easily prove that associated to (27) there is a unique solution $y^{\#} \geq 1$ that does not depend on λ_0 and $\underline{c}$. Given $y^{\#}$ equations (26) and (27) become a non linear system of two equations for two variables λ_0 and $\underline{c}$:

$$\lambda_0 = y^{\#}(m - \underline{c}n)(1 - \theta), \qquad (28)$$

$$\underline{c} \left[\frac{a - \rho}{a(2a - \rho)} - \frac{1}{a} \left(y^{\#}\right)^{\frac{a}{a-\rho}} \right] = \frac{m}{na} \left(\frac{\rho - a}{\rho - 2a} \right) - x_0 \left(y^{\#}\right)^{\frac{a}{a-\rho}} . \qquad (29)$$

Note that the second equation can be solved independently to give $\underline{c}$. This result can then be plugged into the first equation in order find λ_0.

Equation (26) shows clearly that $\underline{c} < m/n = c^b$ when $\underline{c} < ax_0 < c^b$. Since $\lambda_0 > 0$ optimal consumption is continuous and not decreasing in t

$$c(t) = \begin{cases} \underline{c} & \text{if} \quad t \leq T \\ c^b - \frac{\lambda(t)}{n(1-\theta)} & \text{if} \quad t \geq T. \end{cases}$$

Dependence of $\underline{c}$ on x_0. Since $y^\# > 1$, we have $\left(y^\#\right)^{a/(a-\rho)} > (\rho - a)/(\rho - 2a)$. Hence in the r.h.s. of (29) the factor of $\underline{c}$ is negative. Thus (for $x_0 < x^b$), a greater x_0 results in a greater $\underline{c}$:

$$\frac{d\underline{c}}{dx_0} > 0.$$

Dependence of $\underline{c}$ on θ. To compute $\frac{d\underline{c}}{d\theta}$ we must first compute $\frac{dy^\#}{d\theta}$. Deriving (27) with respect to θ we have that

$$\frac{dy^\#}{d\theta} = \frac{a}{(1-\theta)^2 \left(1 - (y^\#)^{\frac{a}{\rho - a}}\right)}.$$

As $y^\# > 1$ and $a > \rho$, we have $\frac{dy^\#}{d\theta} > 0$. Using equation (29) we can find

$$\frac{d\underline{c}}{d\theta} = -\frac{\left(y^\#\right)^{\frac{\rho}{a-\rho}} a^2 \frac{dy^\#}{d\theta} A(x_0 na - m)}{n(Aa - (y^\#)^{\frac{\rho}{a-\rho}})^2 (a-\rho) y^\#} > 0 \ ,$$

where $A = \frac{a-\rho}{a(2a-\rho)}$.

Dependence of T on θ. Substituting λ_0 in (25) by its expression in (28), one finds:

$$T = \frac{1}{a-\rho} \ln y^\# \ ,$$

which is an increasing function of θ since $dy^\#/d\theta > 0$.

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